ON THE VANISHING RANGES FOR THE COHOMOLOGY OF FINITE GROUPS OF LIE TYPE

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Abstract. Let $G(\mathbb{F}_q)$ be a finite Chevalley group defined over the field of $q = p^r$ elements, and k be an algebraically closed field of characteristic $p > 0$. A fundamental open and elusive problem has been the computation of the cohomology ring $\mathrm{H}^\bullet(G(\mathbb{F}_q), k)$. In this paper we determine initial vanishing ranges which improves upon known results. For root systems of type A_n and C_n, the first non-trivial cohomology classes are determined when p is larger than the Coxeter number (larger than twice the Coxeter number for type A_n with $n > 1$ and $r > 1$). In the process we make use of techniques involving line bundle cohomology for the flag variety G/B and its relation to combinatorial data from Kostant Partition Functions.

1 Introduction

1.1 Let G be a simple algebraic group over an algebraically closed field k of prime characteristic $p > 0$ which is split over the prime field $\mathbb{F}_p$. Let $\mathrm{Fr} : G \to G$ denote the Frobenius map and set $q = p^r$. The fixed points of the rth iterate of the Frobenius map, denoted $G(\mathbb{F}_q)$, is a finite Chevalley group. An outstanding open problem of major interest for algebraists and topologists has been to determine the cohomology ring $\mathrm{H}^\bullet(G(\mathbb{F}_q), k)$[1]. In 2005, during a talk at an Oberwolfach conference, Friedlander mentioned that so little is known about this computation that it is not even known in which degree the first non-trivial cohomology class occurs.

Our paper aims to address this fundamental question by investigating two problems:

(1.1.1) Determining Vanishing Ranges: Finding $D > 0$ such that the cohomology groups $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ for $0 < i < D$.

(1.1.2) Locating the First Non-Trivial Cohomology Class: In many instances in conjunction with the aforementioned problem, we will find a D such that $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ for $0 < i < D$ and $\mathrm{H}^D(G(\mathbb{F}_q), k) \neq 0$. A D satisfying this property will be called a *sharp bound*.

There have been earlier results in the 1970s and 80s addressing (1.1.1). Quillen [Q] showed that $\mathrm{H}^i(GL_n(\mathbb{F}_q), k) = 0$ for all $0 < i < r(p-1)$ and all n. In that work, he noted

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[1] In the cross-characteristic situation (i.e., if char $k = l$ and $\gcd(l, p) = 1$) much more is known about the cohomology of $\mathrm{H}^\bullet(G(\mathbb{F}_q), k)$ (cf. [AM, Chapter VII]). In fact in many cases in the non-describing characteristic the cohomology (including the ring structure) is completely determined.

that the arguments showed for any G as above, there exists a constant C depending on the root system such that $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ for $0 < i < r \cdot C$. However, no explicit value of C is given except for $G = SL_2$ (and p odd) in which case one can take $C = (p-1)/2$. Furthermore, it was not determined whether these vanishing ranges were sharp. Indeed, in the case of SL_2, one can see from work of Carlson [C] that these bounds are not sharp in general. Quillen's original work arose in the context of certain K-theory computations. Friedlander [F] later used K-theoretic techniques to find vanishing ranges for more general reductive groups. Later work of Hiller [H] extended Friedlander's result and found vanishing ranges for groups of all types.

Friedlander and Parshall [FP, (A.1) Lemma] found a sharp bound for the Borel subgroup $B(\mathbb{F}_q)$ of $GL_n(\mathbb{F}_q)$. Independent of this work, Barbu [B] constructed a non-zero cohomology class in $\mathrm{H}^{2p-2}(GL_n(\mathbb{F}_p), k)$ for $p \geq n$. In this paper he conjectured that the sharp bound is $D = 2p - 3$ for $GL_n(\mathbb{F}_p)$ when $n \geq 2$ and $p \geq 3$ (cf. [B, Section 1, Conjecture 4.11]). Since that time, few if any results have been obtained in this direction.

1.2 The strategy in addressing (1.1.1) and (1.1.2) will entail using new and powerful techniques developed by the authors which relate $\mathrm{H}^i(G(\mathbb{F}_q), k)$ to extensions over G via a truncated version of the induction functor (cf. [BNP1, BNP2, BNP3, BNP5, BNP6]). An outline of the overall strategy is presented in the diagram below. For the purposes in this paper we will use a non-truncated induction functor $\mathcal{G}_r(-)$. We demonstrate that when applied to the trivial module k, $\mathcal{G}_r(k)$ has a filtration with factors of the form $H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}$ (cf. Proposition 2.4). The G-cohomology of these factors can be analyzed by using the Lyndon-Hochschild-Serre (LHS) spectral sequence involving the first Frobenius kernel G_r (cf. Section 3.1). In particular for $r = 1$, we can apply the results of Kumar-Lauritzen-Thomsen [KLT] to bound the dimension of the cohomology group $\mathrm{H}^\bullet(G(\mathbb{F}_p), k)$ from above (cf. Theorem 3.3). The upper bound on the dimension involves the combinatorics of the well-studied Kostant Partition Function. This reduces the question of the vanishing of the finite group cohomology to a question involving the combinatorics of the underlying root system Φ.

$$
\begin{array}{ccccc}
 & \text{Induction Functor} & & & \\
\mathrm{H}^i(G(\mathbb{F}_q), k) & \Longrightarrow & \mathrm{H}^i(G, \mathcal{G}_r(k)) & & \\
 & & \Downarrow \ \text{Filtrations} & & \\
 & & \mathrm{H}^i(G, H^0(\lambda) \otimes H(\lambda^*)^{(r)}) \Longrightarrow \mathrm{H}^i(G_1, H^0(\lambda)) & \Longrightarrow & \text{Root Combinatorics.} \\
 & & \text{LHS Spectral Sequences} & \text{Kostant Partition Functions} &
\end{array}
$$

More specifically, for a group G of classical type and $\mathrm{H}^i(G(\mathbb{F}_q), k)$, under the assumption that $p > h$ (the Coxeter number), in Theorem 4.4, we identify a vanishing range which

improves upon (in almost all cases) the ranges of [H]. Furthermore, in type C_n and A_n, we identify a *sharp vanishing bound* which addresses (1.1.2) (cf. Theorems 5.4, 6.13, 6.14). These bounds are established for primes larger than the Coxeter number, except for type A_n with $r > 1$ where sharp vanishing bounds are found for p greater than twice the Coxeter number. Finally, as a demonstration of the effectiveness of our methods we verify Barbu's Conjecture for $G = GL_n(\mathbb{F}_q)$ when $n \geq 2$ and $p \geq n+2$ (cf. Theorem 6.15).

Our results provide a conceptual description of how the geometry of the nilpotent cone plays a role in the description of the cohomology $\mathrm{H}^i(G(\mathbb{F}_p), k)$. In particular we prove for $p > h$ (cf. Theorem 3.3):

$$\dim \mathrm{H}^i(G(\mathbb{F}_p), k) \leq \sum_{\{w\in W \mid \ell(w)\equiv i \bmod 2\}} \sum_{\mu\in X(T)_+} \sum_{u\in W} (-1)^{\ell(u)} P_{\frac{i-\ell(w)}{2}}(u\cdot(p\mu + w\cdot 0) - \mu).$$

Here, $P_n(-)$ is the Kostant Partition Function. The root combinatorics involving the Kostant Partition Function naturally arises in the context of composition factor multiplicities in the ring of regular functions on the nilpotent cone $\mathcal{N}$ of $\mathfrak{g} = \mathrm{Lie}\ G$ (cf. [J2] [Br]). This result reinforces work of Carlson, Lin and Nakano [CLN] where they prove that the spectrum of this cohomology ring is given by the coordinate algebra on $\mathcal{N}^{\mathbb{F}_p}/G(\mathbb{F}_p)$ where $\mathcal{N}^{\mathbb{F}_p}$ is the variety inside of $\mathcal{N}$ consisting of $\mathbb{F}_p$-expressible elements.

The sections of the paper are outlined as follows. In Section 2, we review our previous work and develop the necessary cohomological tools related to induction functors which will be used to determine vanishing ranges. In Section 3, we present some further cohomological properties relating extensions over G with those over the Frobenius kernel G_r. In Section 4, our general vanishing bounds are presented. Finally, Sections 5 and 6 deal with the special cases of root systems of types C_n and A_n respectively. The reader might be surprised to see type C treated prior to type A. It turns out that the root systems of type C are by far the easiest to be dealt with. The authors in future work plan to address (1.1.1) and (1.1.2) in the case of the remaining classical groups and the exceptional groups.

2 Relating $G(\mathbb{F}_q)$ and G

2.1 Notation. Throughout this paper, we will follow the basic conventions provided in [J1]. Let G be a simple simply connected algebraic group scheme which is defined and split over the finite field $\mathbb{F}_p$ with p elements, and let k be a field of characteristic p. For $r \geq 1$, let $G_r := \ker F^r$ be the rth Frobenius kernel of G and $G(\mathbb{F}_q)$ be the associated finite Chevalley group. Let T be a maximal split torus and Φ be the root system associated to (G, T). The positive (resp. negative) roots are Φ^+ (resp. Φ^-), and Δ is the set of simple roots. Let B be a Borel subgroup containing T corresponding to the negative roots and U be the unipotent radical of B. For a given root system of rank n, the simple roots will be denoted by $\alpha_1, \alpha_2, \ldots, \alpha_n$. We will adhere to the Bourbaki ordering of simple roots. In particular, for type B_n, α_n denotes the unique short simple root and for type C_n, α_n denotes the unique long simple root. The longest (positive) root will be denoted $\tilde{\alpha}$, and for root systems with multiple root lengths, the longest short root will be denoted α_0. Let W denote the Weyl group associated to Φ, and, for $w \in W$, let $\ell(w)$ denote the length of the word.

Let $\mathbb{E}$ be the Euclidean space associated with Φ, and the inner product on $\mathbb{E}$ will be denoted by $\langle\ ,\ \rangle$. Let $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$ be the coroot corresponding to $\alpha \in \Phi$. In this case, the fundamental weights (basis dual to $\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee$) will be denoted by $\omega_1, \omega_2, \dots, \omega_n$. Let $X(T)$ be the integral weight lattice spanned by these fundamental weights. The set of dominant integral weights is denoted by $X(T)_+$. For a weight $\lambda \in X(T)$, set $\lambda^* := -w_0\lambda$ where w_0 is the longest word in the Weyl group W. By $w \cdot \lambda := w(\lambda+\rho) - \rho$ we denote the "dot" action of W on $X(T)$, with ρ being the half-sum of the positive roots. For $\alpha \in \Delta$, $s_\alpha \in W$ denotes the reflection in the hyperplane determined by α.

For a G-module M, let $M^{(r)}$ be the module obtained by composing the underlying representation for M with F^r. Moreover, let M^* denote the dual module. For $\lambda \in X(T)_+$, let $H^0(\lambda) := \operatorname{ind}_B^G \lambda$ be the induced module and $V(\lambda) := H^0(\lambda^*)^*$ be the Weyl module of highest weight λ.

2.2 We record two observations on roots that will be used at various points in the paper:

Observation (A)**.** If $\beta \in \Phi^+$ with $\beta \neq \tilde{\alpha}$, then $\langle \beta, \tilde{\alpha}^\vee\rangle \in \{0,1\}$.

Observation (B)**.** If $w \in W$ admits a reduced expression $w = s_{\beta_1}s_{\beta_2}\dots s_{\beta_m}$ with $\beta_i \in \Delta$ and $m = \ell(w)$, then

$$-w\cdot 0 = \beta_1 + s_{\beta_1}(\beta_2) + s_{\beta_1}s_{\beta_2}(\beta_3) + \dots + s_{\beta_1}s_{\beta_2}\dots s_{\beta_{m-1}}(\beta_m).$$

Moreover, this is the unique way in which $-w\cdot 0$ can be expressed as a sum of distinct positive roots.

2.3 The Induction Functor. Set $\mathcal{G}_r(k) := \operatorname{ind}_{G(\mathbb{F}_q)}^G(k)$. While this G-module is infinite dimensional, it provides a potential way to relate extensions over $G(\mathbb{F}_q)$ with extensions over G. Indeed, by Generalized Frobenius Reciprocity and the fact that $G/G(\mathbb{F}_q)$ is affine, we have the following key observation.

Proposition . *Let M, N be rational G-modules. Then, for all $i \geq 0$,*

$$\operatorname{Ext}^i_{G(\mathbb{F}_q)}(M,N) \cong \operatorname{Ext}^i_G(M, N\otimes \mathcal{G}_r(k)).$$

2.4 Good Filtrations. To make the desired computations of cohomology groups, we will make use of Proposition 2.3 (with $M = k = N$). In addition, we will use a special filtration on $\mathcal{G}_r(k)$. Recall that a G-module M has a *good filtration* if it admits a filtration with successive quotients of the form $H^0(\lambda)$, $\lambda \in X(T)_+$ [J1, II 4.16].

One may consider $k[G]$ as a $G\times G$-module via the left and right regular actions, respectively. A result due to Donkin and Koppinen [J1, II 4.20] now says that $k[G]$ as a $G\times G$-module admits a good filtration with factors of the form $H^0(\lambda)\otimes H^0(\lambda^*)$, each $\lambda \in X(T)_+$ appearing exactly once. Here $(g_1, g_2) \in G\times G$ acts via $g_1 \otimes g_2$ on each factor. If one takes the diagonal embedding of G into $G \times G$ one can use this fact to show that $k[G]$ admits a good filtration under the adjoint action of G.

For our purposes, we modify this slightly by using a partial Frobenius twist. Consider the composite

$$\phi : G \overset{\text{diag}}{\to} G\times G \overset{1\times F^r}{\to} G \times G.$$

That is, when we take the diagonal embedding, we apply the Frobenius morphism r-times to the second factor giving $\phi(g) = (g, F^r(g))$. Let $G \times G$ act on $k[G]$ via the left and right regular representations as above, and then restrict this to a module over G via ϕ. Denote the resulting G-module by $k[G]^\vee$. The next proposition investigates filtrations on $k[G]^\vee$.

Proposition . *As G-modules $\mathcal{G}_r(k) \cong k[G]^\vee$. Moreover, $\mathcal{G}_r(k)$ has a filtration with factors of the form $H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}$ with multiplicity one for each $\lambda \in X(T)_+$.*

Proof. Let $f \in k[G]$ and $g, x \in G$. If we denote the action of G on $k[G]^\vee$ by $\star$ then

$$(g \star f)(x) = f(g^{-1} x F^r(g)).$$

Next we define a Lang map $L : G \to G$ via $L(g) = gF^r(g^{-1})$. By setting $L^*(f) = f \circ L$ one obtains a bijection ([Hum2, 1.4])

$$L^* : k[G] \to \mathcal{G}_r(k) = \operatorname{ind}_{G(\mathbb{F}_q)}^G(k) = \{f \in k[G] \mid f(gh) = f(g) \text{ for all } g \in G, h \in G(\mathbb{F}_q)\}.$$

Observe that

$$L^*(g \star f)(x) = (g \star f)(L(x)) = f(g^{-1} L(x) F^r(g))$$

while

$$\begin{aligned} (g(L^*(f)))(x) &= L^*(f)(g^{-1}x) = f(L(g^{-1}x)) = f(g^{-1}xF^r(x^{-1}g)) \\ &= f(g^{-1}xF^r(x^{-1})Fr^r(g)) = f(g^{-1}L(x)F^r(g)). \end{aligned}$$

Hence, L^* is a G-equivariant bijective map from $k[G]^\vee$ onto $\mathcal{G}_r(k)$.

Since $k[G]$, viewed as a $G \times G$-module, has a good filtration with factors $H^0(\lambda) \otimes H^0(\lambda^*)$ and $\phi(g)$ acts on each factor via $g \otimes F^r(g)$, it follows that $k[G]^\vee$ has a filtration with factors of the form $H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}$ and multiplicity one for each $\lambda \in X(T)_+$, as claimed. □

2.5 Given weights $\lambda, \mu \in X(T)$, recall that we say $\mu < \lambda$ if and only if $\lambda - \mu = \sum_{\alpha \in \Delta} c_\alpha \alpha$ for integers $c_\alpha \geq 0$. That is, $\lambda - \mu$ must lie in the positive root lattice. In addition we say two weights λ and μ are linked if there exists an element w of the affine Weyl group such that $\mu = w \cdot \lambda$. Note that two weights $(\lambda_1, \lambda_2), (\mu_1, \mu_2) \in X(T \times T)$ are linked for the algebraic group $G \times G$ if and only if the components are linked for G.

For each dominant weight λ we define two finite saturated sets, namely

$$\pi_{<\lambda} = \{\mu \in X(T)_+ | \ \mu < \lambda\} \text{ and } \pi_{\leq\lambda} = \{\mu \in X(T)_+ | \ \mu \leq \lambda\}.$$

According to [J1, II A.15] the $G \times G$-module $k[G]$ has two submodules $M_{<\lambda}$ and $M_{\leq\lambda}$, both admitting good filtrations with factors $H^0(\nu) \otimes H^0(\nu^*)$ where each $\nu \in \pi_{<\lambda}$ ($\pi_{\leq\lambda}$, respectively) appears exactly once. By $S_{<\lambda}$ and $S_{\leq\lambda}$ we denote the $G \times G$-summands of $M_{<\lambda}$ and $M_{\leq\lambda}$, respectively, whose $G \times G$-composition factors have highest weights contained in the same $G \times G$-linkage class as (λ, λ^*). Similarly we define the quotients $Q_{\not<\lambda} = k[G]/S_{<\lambda}$ and $Q_{\not\leq\lambda} = k[G]/S_{\leq\lambda}$.

The group G acts on these modules via the embedding ϕ. From Proposition 2.4 and [J1, II 4.17] one obtains the following result.

Theorem . *For each $\lambda \in X(T)_+$, there exist short exact sequences of G-modules*

$$0 \to S_{<\lambda} \to \mathcal{G}_r(k) \to Q_{\not<\lambda} \to 0$$

and

$$0 \to S_{\leq\lambda} \to \mathcal{G}_r(k) \to Q_{\nleq\lambda} \to 0$$

with the following properties:

(a) $S_{<\lambda}$ $(S_{\leq\lambda})$ *has a filtration with factors of the form* $H^0(\nu) \otimes H^0(\nu^*)^{(r)}$ *where* $\nu < \lambda$ $(\nu \leq \lambda)$ *and* ν *is linked to* λ.

(b) $Q_{\nless\lambda}$ $(Q_{\nleq\lambda})$ *has a filtration with factors of the form* $H^0(\nu) \otimes H^0(\nu^*)^{(r)}$ *where* $\nu \nless \lambda$ $(\nu \nleq \lambda)$ *or* ν *is not linked to* λ.

(c) *The multiplicity in all cases is one.*

2.6 Upper Bounds for $\mathrm{Ext}^i_{G(\mathbb{F}_q)}$ and Vanishing Criteria. The next theorem and its corollaries illustrate the usefulness of the existence of the filtrations in Proposition 2.4.

Theorem . *Let* M, N *be rational* G*-modules and* $i \geq 0$*. Then*

$$\dim \mathrm{Ext}^i_{G(\mathbb{F}_q)}(M,N) \leq \sum_{\lambda \in X(T)_+} \dim \mathrm{Ext}^i_G(M, N \otimes H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}).$$

Proof. The result follows immediately from Proposition 2.3, Proposition 2.4 and the long exact sequence in cohomology associated to a short exact sequence. □

One obtains the following vanishing criterion.

Corollary (A)**.** *Let* M, N *be rational* G*-modules and* $i \geq 0$*. If* $\mathrm{Ext}^i_G(M, N \otimes H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) = 0$ *for all* $\lambda \in X(T)_+$*, then* $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(M,N) = 0$.

In particular, in the special case of $M = k = N$, we get the following criterion for a vanishing range in cohomology. We will see later how to identify such an m for a given G.

Corollary (B)**.** *Let* m *be the least positive integer such that there exists* $\lambda \in X(T)_+$ *with* $\mathrm{H}^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$*. Then* $\mathrm{H}^i(G(\mathbb{F}_q), k) \cong \mathrm{H}^i(G, \mathcal{G}_r(k)) = 0$ *for* $0 < i < m$.

2.7 Non-vanishing. While the identification of an m satisfying Corollary 2.6(B) gives a vanishing range, it does not a priori follow that $\mathrm{H}^m(G(\mathbb{F}_q), k) \neq 0$. In this section, we develop some conditions under which this conclusion could be made, as well as conditions under which one might be able to precisely identify the cohomology group. As in Corollary 2.6(A), we have the following.

Proposition (A)**.** *Let* $\lambda \in X(T)_+$ *and* $M \in \{S_{<\lambda}, S_{\leq\lambda}, Q_{\nless\lambda}, Q_{\nleq\lambda}\}$*. Suppose that*

$$\mathrm{H}^i(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)}) = 0$$

for all $\nu \in X(T)_+$ *which appear in the filtration for* M*. Then* $\mathrm{H}^i(G, M) = 0$.

The next proposition reduces the problem of showing the non-vanishing of $\mathrm{H}^m(G, \mathcal{G}_r(k))$ (and hence of $\mathrm{H}^m(G(\mathbb{F}_q), k)$) to showing non-vanishing for a submodule of $\mathcal{G}_r(k)$.

Proposition (B)**.** *Let* m *be a positive integer. For any* $\lambda \in X(T)_+$,

(i) *if* $\mathrm{H}^{m+1}(G, S_{<\lambda}) = 0$*, then the map* $\mathrm{H}^m(G, \mathcal{G}_r(k)) \to \mathrm{H}^m(G, Q_{\nless\lambda})$ *is surjective;*

(ii) *if* $\mathrm{H}^m(G, S_{<\lambda}) = 0$*, then the map* $\mathrm{H}^m(G, \mathcal{G}_r(k)) \to \mathrm{H}^m(G, Q_{\nless\lambda})$ *is injective;*

(iii) *if the conditions in (i) and (ii) hold, then* $\mathrm{H}^m(G,\mathcal{G}_r(k)) \cong \mathrm{H}^m(G, Q_{\not<\lambda})$.

Proof. Consider the short exact sequence

$$0 \to S_{<\lambda} \to \mathcal{G}_r(k) \to Q_{\not<\lambda} \to 0$$

and the associated long exact sequence in cohomology

$$\cdots \to \mathrm{H}^m(G, S_{<\lambda}) \to \mathrm{H}^m(G,\mathcal{G}_r(k)) \to \mathrm{H}^m(G, Q_{\not<\lambda}) \to \mathrm{H}^{m+1}(G, S_{<\lambda}) \to \cdots .$$

The claims in (i) and (ii) follow immediately from this exact sequence. □

Remark . In order to show the non-vanishing of $\mathrm{H}^m(G(\mathbb{F}_q), k)$, condition (i) is the crucial condition. Whereas condition (ii) potentially allows us to identify $\mathrm{H}^m(G(\mathbb{F}_q), k)$ precisely. Note that condition (ii) is immediately satisfied by any weight λ which is minimal with respect to the above standard ordering such that $\mathrm{H}^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$.

Having reduced the problem to $\mathrm{H}^i(G, Q_{\not<\lambda})$, we make some similar homological observations about this group.

Proposition (C)**.** *Let m be the least positive integer such that there exists $\nu \in X(T)_+$ with $\mathrm{H}^m(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)}) \neq 0$. For any $\lambda \in X(T)_+$,*

(i) *the map* $\mathrm{H}^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \to \mathrm{H}^m(G, Q_{\not<\lambda})$ *is injective;*

(ii) *if* $\mathrm{H}^m(G, Q_{\not\leq\lambda}) = 0$, *then* $\mathrm{H}^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \cong \mathrm{H}^m(G, Q_{\not<\lambda})$.

Proof. Consider the short exact sequence

$$0 \to H^0(\lambda) \otimes H^0(\lambda^*)^{(r)} \to Q_{\not<\lambda} \to Q_{\not\leq\lambda} \to 0$$

and the associated LES

$$\cdots \to \mathrm{H}^{m-1}(G, Q_{\not\leq\lambda}) \to \mathrm{H}^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \to \mathrm{H}^m(G, Q_{\not<\lambda}) \to \mathrm{H}^m(G, Q_{\not\leq\lambda}) \to \cdots .$$

If $m > 1$, the claims follow immediately since the first term is zero by minimality of m.

Suppose that $m = 1$. Suppose first that λ is linked to the zero weight. Since we certainly cannot have $0 > \lambda$, by Theorem 2.5(b), $\mathrm{H}^0(G, Q_{\not\leq\lambda}) = \mathrm{Hom}_G(k, Q_{\not\leq\lambda}) = 0$, and the argument follows as above.

Suppose now that λ is not linked to the zero weight. The module $Q_{\not<\lambda}$ may be decomposed as a direct sum $M_1 \oplus M_2$ where M_1 has a filtration with factors of the form $H^0(\nu) \otimes H^0(\nu^*)^{(1)}$ with ν linked to λ and M_2 has such a filtration with ν not linked to λ. Then $\mathrm{H}^1(G, Q_{\not<\lambda}) = \mathrm{H}^1(G, M_1) \oplus \mathrm{H}^1(G, M_2)$. Consider the short exact sequence

$$0 \to H^0(\lambda) \otimes H^0(\lambda^*)^{(1)} \to M_1 \to M_1' \to 0, \tag{2.7.1}$$

which defines a module M_1' with a corresponding filtration. Observe that $Q_{\not\leq\lambda} \cong M_1' \oplus M_2$, and hence $\mathrm{H}^1(G, Q_{\not\leq\lambda}) \cong \mathrm{H}^1(G, M_1') \oplus \mathrm{H}^1(G, M_2)$. Since λ is not linked to zero, $\mathrm{H}^0(G, M_1') = \mathrm{Hom}_G(k, M_1') = 0$. Arguing as above with the LES associated to (2.7.1) gives

$$\mathrm{H}^1(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \hookrightarrow \mathrm{H}^1(G, M_1) \hookrightarrow \mathrm{H}^1(G, Q_{\not<\lambda}),$$

and so part (i) holds. For part (ii), it follows from the additional assumption that $\mathrm{H}^1(G, M_1') = 0$ and $\mathrm{H}^1(G, M_2) = 0$. Again using the LES associated to (2.7.1), it follows that

$$\mathrm{H}^1(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \cong \mathrm{H}^1(G, M_1) = \mathrm{H}^1(G, M_1) \oplus \mathrm{H}^1(G, M_2) \cong \mathrm{H}^1(G, Q_{\not<\lambda})$$

as claimed.

□

2.8 Combining the propositions in the preceding section, we can obtain a condition under which we have sharp vanishing bounds, and an explicit identification of $\mathrm{H}^m(G(\mathbb{F}_q), k)$ with a single G-cohomology group.

Theorem (A). *Let m be the least positive integer such that there exists $\nu \in X(T)_+$ with $\mathrm{H}^m(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)}) \neq 0$. Let $\lambda \in X(T)_+$ be such that $\mathrm{H}^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$. Suppose $\mathrm{H}^{m+1}(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)}) = 0$ for all $\nu < \lambda$ that are linked to λ. Then*

(i) *$\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ for $0 < i < m$;*
(ii) *$\mathrm{H}^m(G(\mathbb{F}_q), k) \neq 0$;*
(iii) *if, in addition, $\mathrm{H}^m(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)}) = 0$ for all $\nu \in X(T)_+$ with $\nu \neq \lambda$, then $\mathrm{H}^m(G(\mathbb{F}_q), k) \cong \mathrm{H}^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)})$.*

Proof. Part (i) follows from Corollary 2.6(B). For part (ii), note first that, by the hypothesis on λ and Proposition 2.7(C)(i), $\mathrm{H}^m(G, Q_{\not<\lambda}) \neq 0$. On the other hand, by the hypothesis on weights less than λ and Proposition 2.7(A), $\mathrm{H}^{m+1}(G, S_{<\lambda}) = 0$. Hence, by Proposition 2.7(B)(i), the map $\mathrm{H}^m(G, \mathcal{G}_r(k)) \to \mathrm{H}^m(G, Q_{\not<\lambda})$ is surjective. Therefore, by Proposition 2.3, $\mathrm{H}^m(G(\mathbb{F}_q), k) \cong \mathrm{H}^m(G, \mathcal{G}_r(k)) \neq 0$ as claimed.

For part (iii), by the added assumption and Proposition 2.7(A), we have $\mathrm{H}^m(G, S_{<\lambda}) = 0$ and $\mathrm{H}^m(G, Q_{\not\leq\lambda}) = 0$. By Proposition 2.7(B)(iii) and Proposition 2.7(C)(ii), we have $\mathrm{H}^m(G, \mathcal{G}_r(k)) \cong \mathrm{H}^m(G, Q_{\not<\lambda}) \cong \mathrm{H}^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)})$ as claimed. □

From the filtration on $\mathcal{G}_r(k)$ in Proposition 2.4, $\mathrm{H}^i(G(\mathbb{F}_q), k) \cong \mathrm{H}^i(G, \mathcal{G}_r(k))$ can be decomposed as a direct sum over linkage classes of dominant weights. As such, using an analogous argument, a slightly weaker condition for non-vanishing can be obtained.

Theorem (B). *For a fixed linkage class $\mathcal{L}$, let m be the least positive integer such that there exists $\nu \in \mathcal{L}$ with $\mathrm{H}^m(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)}) \neq 0$. Let $\lambda \in \mathcal{L}$ be such that $\mathrm{H}^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$. Suppose $\mathrm{H}^{m+1}(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)}) = 0$ for all $\nu < \lambda$ in $\mathcal{L}$. Then $\mathrm{H}^m(G(\mathbb{F}_q), k) \neq 0$.*

3 Properties of the Cohomology Groups

In Section 2, it was shown that knowledge of cohomology groups of the form $\mathrm{H}^i(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)})$ for $\nu \in X(T)_+$ provides information on $\mathrm{H}^i(G(\mathbb{F}_q), k)$. In this section, we study these G-cohomology groups and collect a number of useful properties that will be used throughout the remainder of the paper.

3.1 Reducing to G_r-cohomology. We will make frequent use of the following identification of G-extensions with G_r-cohomology.

Lemma . *Let $\nu_1, \nu_2 \in X(T)_+$. Assume that $\mathrm{H}^j(G_r, H^0(\nu_1))^{(-r)}$ admits a good filtration for all $j > 0$. Then for all j*

$$\mathrm{H}^j(G, H^0(\nu_1) \otimes H^0(\nu_2^*)^{(r)}) \cong \mathrm{Ext}^j_G(V(\nu_2)^{(r)}, H^0(\nu_1)) \cong \mathrm{Hom}_G(V(\nu_2), \mathrm{H}^j(G_r, H^0(\nu_1))^{(-r)}).$$

Proof. The first isomorphism is immediate. For the crucial second isomorphism, consider the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{G/G_r}(V(\nu_2)^{(r)}, \mathrm{H}^j(G_r, H^0(\nu_1))) \Rightarrow \mathrm{Ext}^{i+j}_G(V(\nu_2)^{(r)}, H^0(\nu_1)).$$

We have

$$\begin{aligned} E_2^{i,j} &= \mathrm{Ext}^i_{G/G_r}(V(\nu_2)^{(r)}, \mathrm{H}^j(G_r, H^0(\nu_1))) \\ &\cong \mathrm{Ext}^i_G(V(\nu_2), \mathrm{H}^j(G_r, H^0(\nu_1))^{(-r)}). \end{aligned}$$

For $\nu \in X(T)_+$, $i > 0$, and V a G-module which admits a good filtration, we have $\mathrm{Ext}^i_G(V(\nu), V) = 0$ (cf. [J1, Prop. II 4.16]). By the hypothesis, we conclude that $E_2^{i,j} = 0$ for all $i > 0$ and the spectral sequence collapses to a single vertical column. This implies that $E_2^{0,j} \cong \mathrm{Ext}^j_G(V(\nu_2)^{(r)}, H^0(\nu_1))$ for all j. □

The assumption that $\mathrm{H}^j(G_r, H^0(\nu))^{(-r)}$ admits a good filtration is a long-standing conjecture of Donkin. For $p > h$ (the Coxeter number of the root system associated to G), this is known for $r = 1$ by results of Andersen-Jantzen [AJ] and Kumar-Lauritzen-Thomsen [KLT]. For arbitrary r, this is known only for all degrees in the case $G = SL_2$. When r is arbitrary and $i = 1$ (p arbitrary) or $i = 2$ ($p \geq 3$), Bendel-Nakano-Pillen verified the assumption by direct computation [BNP4, BNP7]. Wright [W] has recently verified the $p = 2$, $i = 2$ case.

We will apply Lemma 3.1 at several points in the $r = 1$ case for direct applications to $G(\mathbb{F}_p)$ as well as inductively for dealing with $G(\mathbb{F}_q)$. As such, we will generally assume for the remainder of the paper that $p > h$.

3.2 Dimensions for $r = 1$. From Lemma 3.1, to obtain information about $\mathrm{H}^i(G, H^0(\nu) \otimes H^0(\nu^*)^{(1)})$ for $\nu \in X(T)_+$, it suffices to consider $\mathrm{Hom}_G(V(\nu), \mathrm{H}^i(G_1, H^0(\nu)^{(-1)})$. It is well-known that, from block considerations, $\mathrm{H}^i(G_1, H^0(\nu)) = 0$ unless $\nu = w \cdot 0 + p\mu$ for $w \in W$ and $\mu \in X(T)$. For $p > h$, from [AJ] and [KLT], we have

$$\mathrm{H}^i(G_1, H^0(\nu))^{(-1)} = \begin{cases} \mathrm{ind}_B^G(S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*) \otimes \mu) & \text{if } \nu = w \cdot 0 + p\mu \\ 0 & \text{otherwise,} \end{cases} \tag{3.2.1}$$

where $\mathfrak{u} = \mathrm{Lie}(U)$. Note also that, since $p > h$ and ν is dominant, μ must also be dominant.

For a weight ν and $n \geq 0$, let $P_n(\nu)$ denote the dimension of the ν-weight space of $S^n(\mathfrak{u}^*)$. Equivalently, for $n > 0$, $P_n(\nu)$ denotes the number of times that ν can be expressed as a sum of exactly n positive roots, while $P_0(0) = 1$. The function P_n is often referred to as *Kostant's Partition Function*. By using [AJ, 3.8], [KLT, Thm 2], Lemma 3.1, and (3.2.1), we can give an explicit formula for the dimension of $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)})$. Namely,

Proposition . *Assume $p > h$. Let $\lambda = p\mu + w \cdot 0 \in X(T)_+$. Then*

$$\dim \mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = \sum_{u \in W} (-1)^{\ell(u)} P_{\frac{i-\ell(w)}{2}}(u \cdot \lambda - \mu).$$

3.3 From Proposition 2.3 and Proposition 2.4, we can now deduce the following upper bound on the dimensions of the cohomology groups $\mathrm{H}^i(G(\mathbb{F}_q), k)$.

Theorem . *Assume $p > h$.*

$$\dim \mathrm{H}^i(G(\mathbb{F}_p), k) \leq \sum_{\{w \in W \mid \ell(w) \equiv i \bmod 2\}} \sum_{\mu \in X(T)_+} \sum_{u \in W} (-1)^{\ell(u)} P_{\frac{i-\ell(w)}{2}}(u \cdot (p\mu + w \cdot 0) - \mu).$$

3.4 Degree Bounds. From our discussion in Section 2, to find vanishing ranges for $\mathrm{H}^\bullet(G(\mathbb{F}_p), k)$ (or $\mathrm{H}^\bullet(G(\mathbb{F}_q), k)$ more generally), a first step is to try to identify the least positive i such that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)})$ is non-zero.

Assume that $p > h$ and $\lambda \in X(T)_+$ with $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ for some $i > 0$. From Lemma 3.1 and the discussion in Section 3.2, we know that $\lambda = p\mu + w \cdot 0$ for $w \in W$ and $\mu \in X(T)_+$. Observe that if $\mu = 0$, in order for λ to be dominant, we must have $\lambda = 0$. But, $\mathrm{H}^i(G, k) = 0$ for $i > 0$. Since we are interested in cohomology in non-zero degrees, we may safely assume that $\lambda, \mu \neq 0$. Corollary 3.5 below gives a relationship between i and the weight λ. We first derive a more general relationship that will be useful in inductive arguments.

Proposition . *Assume that $p > h$. Let $\gamma_1, \gamma_2 \in X(T)_+$, both non-zero, such that $\gamma_j = p\delta_j + w_j \cdot 0$ with $\delta_j \in X(T)_+$ and $w_j \in W$ for $j = 1, 2$. Assume $\mathrm{Ext}^i_G(V(\gamma_2)^{(1)}, H^0(\gamma_1)) \neq 0$.*

(a) *Let $\sigma \in \Phi^+$. If Φ is of type G_2, assume further that σ is a long root. Then $p\langle \delta_2, \sigma^\vee \rangle - \langle \delta_1, \sigma^\vee \rangle + \ell(w_1) + \langle w_2 \cdot 0, \sigma^\vee \rangle \leq i$.*

(b) *Let $\tilde{\alpha}$ denote the longest root in Φ^+. Then $p\langle \delta_2, \tilde{\alpha}^\vee \rangle - \langle \delta_1, \tilde{\alpha}^\vee \rangle + \ell(w_1) - \ell(w_2) - 1 \leq i$. Equality requires that $\gamma_2 - \delta_1 = ((i - l(w_1))/2)\tilde{\alpha}$ and $\langle -w_2 \cdot 0, \tilde{\alpha}^\vee \rangle = \ell(w_2) + 1$.*

Proof. By Lemma 3.1, (3.2.1), and Frobenius reciprocity, we have

$$\begin{aligned}
\mathrm{Ext}^i_G(V(\gamma_2)^{(1)}, H^0(\gamma_1)) &\cong \mathrm{Hom}_G(V(\gamma_2), \mathrm{H}^i(G_1, H^0(\gamma_1))^{(-1)}) \\
&\cong \mathrm{Hom}_G(V(\gamma_2), \mathrm{ind}_B^G(S^{\frac{i-\ell(w_1)}{2}}(\mathfrak{u}^*) \otimes \delta_1)) \\
&\cong \mathrm{Hom}_B(V(\gamma_2), S^{\frac{i-\ell(w_1)}{2}}(\mathfrak{u}^*) \otimes \delta_1).
\end{aligned}$$

Since this is non-zero, $\gamma_2 = p\delta_2 + w_2 \cdot 0$ must be a weight of $S^{\frac{i-\ell(w_1)}{2}}(\mathfrak{u}^*) \otimes \delta_1$. In other words, $\gamma_2 - \delta_1 = p\delta_2 - \delta_1 + w_2 \cdot 0$ must be a weight of $S^{\frac{i-\ell(w_1)}{2}}(\mathfrak{u}^*)$.

The vector space $\mathfrak{u}^*$ has a basis of root vectors corresponding to positive roots. So a homogeneous weight of $S^j(\mathfrak{u}^*)$ is a sum of j not necessarily distinct positive roots. Therefore, $\gamma_2 - \delta_1$ must be expressible as a sum of $\frac{i-\ell(w_1)}{2}$ positive roots. For any positive roots σ_1, σ_2 (with σ_2 being long if Φ is of type G_2), we have $\langle \sigma_1, \sigma_2^\vee \rangle \leq 2$. Hence, for $\sigma \in \Phi^+$, we have

$$\langle \gamma_2 - \delta_1, \sigma^\vee \rangle \leq \frac{i - \ell(w_1)}{2} * 2 = i - \ell(w_1). \tag{3.4.1}$$

Substituting $\gamma_2 - \delta_1 = p\delta_2 - \delta_1 + w_2 \cdot 0$ gives

$$p\langle \delta_2, \sigma^\vee \rangle - \langle \delta_1, \sigma^\vee \rangle + \langle w_2 \cdot 0, \sigma^\vee \rangle \leq i - \ell(w_1).$$

Part (a) immediately follows.

For part (b), Note that for $\sigma = \tilde{\alpha}$ equality in Equation (3.4.1) can only hold if $\gamma_2 - \delta_1 = ((i - l(w_1))/2)\tilde{\alpha}$. This follows from Observation 2.2(A). In addition, by Observation 2.2(B),

$-w_2 \cdot 0$ can be expressed uniquely as a sum of precisely $\ell(w_2)$ distinct positive roots. Since at most one of those roots can be $\tilde{\alpha}$, and $\langle \tilde{\alpha}, \tilde{\alpha}^\vee \rangle = 2$, by Observation 2.2(A), we have

$$\langle -w_2 \cdot 0, \tilde{\alpha}^\vee \rangle \leq (\ell(w_2) - 1) * 1 + 2 = \ell(w_2) + 1$$

which gives part (b). □

3.5 As a special case of Proposition 3.4 we have the following result.

Corollary . *Assume that $p > h$. Let $\lambda = p\mu + w \cdot 0$ be a non-zero dominant weight with $0 \neq \mu \in X(T)_+$ and $w \in W$. Assume $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$.*

(a) *Let $\sigma \in \Phi^+$. If Φ is of type G_2, assume further that σ is a long root. Then $(p-1)\langle \mu, \sigma^\vee \rangle + \ell(w) + \langle w \cdot 0, \sigma^\vee \rangle \leq i$.*

(b) *Let $\tilde{\alpha}$ denote the longest root in Φ^+. Then $(p-1)\langle \mu, \tilde{\alpha}^\vee \rangle - 1 \leq i$. Equality requires that $\lambda - \mu = ((i - l(w))/2)\tilde{\alpha}$ and $\langle -w \cdot 0, \tilde{\alpha}^\vee \rangle = \ell(w) + 1$.*

Proof. Parts (a) and (b) follow immediately from Propositon 3.4 by taking $\gamma_1 = \lambda = \gamma_2$. □

In the corollary, since μ is a non-zero dominant weight, $\langle \mu, \tilde{\alpha}^\vee \rangle \geq 1$. Hence, we immediately have that $i \geq p - 2$. It follows from Corollary 2.6(B) that $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ for $0 < i < p - 2$. This will follow as a special case of a more general result in the next section.

4 A Minimal Vanishing Range

4.1 In this section, we use the preceding techniques to determine a general vanishing range for $\mathrm{H}^i(G(\mathbb{F}_q), k)$ for $p > h$. We begin with some further extension properties that will be used in the proof.

Lemma . *Assume that $p > h$ and $r > 1$. Let $\lambda, \mu \in X(T)_+$, both non-zero, and $i \geq 0$. If $\mathrm{Ext}^i_G(V(\lambda)^{(r)}, H^0(\mu)) \neq 0$, then there exists a non-zero weight $\gamma \in X(T)_+$ and nonnegative integers k, l such that*

(a) *$i = k + l$,*

(b) *$\mathrm{Ext}^k_G(V(\lambda)^{(r-1)}, H^0(\gamma)) \neq 0$,*

(c) *$\mathrm{Ext}^l_G(V(\gamma)^{(1)}, H^0(\mu)) \neq 0$, and*

(d) *$\gamma = p\delta + w \cdot 0$ for some $w \in W$ and non-zero $\delta \in X(T)_+$.*

Proof. Consider the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{k,l} = \mathrm{Ext}^k_{G/G_1}(V(\lambda)^{(r)}, \mathrm{H}^l(G_1, H^0(\mu))) \Rightarrow \mathrm{Ext}^{k+l}_G(V(\lambda)^{(r)}, H^0(\mu)).$$

The assumptions imply that there exist nonnegative integers k, l with $k + l = i$ and $\mathrm{Ext}^k_{G/G_1}(V(\lambda)^{(r)}, \mathrm{H}^l(G_1, H^0(\mu))) \neq 0$. The G-module $\mathrm{H}^l(G_1, H^0(\mu))^{(-1)}$ has a good filtration. Therefore, there exists a dominant weight γ with

$$\mathrm{Hom}_{G/G_1}(V(\gamma)^{(1)}, \mathrm{H}^l(G_1, H^0(\mu))) \cong \mathrm{Hom}_G(V(\gamma), \mathrm{H}^l(G_1, H^0(\mu))^{(-1)}) \neq 0 \tag{4.1.1}$$

and

$$\mathrm{Ext}^k_G(V(\lambda)^{(r-1)}, H^0(\gamma)) \cong \mathrm{Ext}^k_{G/G_1}(V(\lambda)^{(r)}, H^0(\gamma)^{(1)}) \neq 0. \tag{4.1.2}$$

Now (4.1.2) implies that γ is of the form $\gamma = p\delta + w\cdot 0$ with $w \in W$ and $\delta \in X(T)_+$. By Lemma 3.1 and (4.1.1), $\mathrm{Ext}^l_G(V(\gamma)^{(1)}, H^0(\mu)) \cong \mathrm{Hom}_G(V(\gamma), \mathrm{H}^l(G_1, H^0(\mu))^{(-1)}) \neq 0$. This forces $\delta \neq 0$. Note that the assumptions also force $\mu = p\delta' + w'\cdot 0$ for some $w' \in W$ and nonzero $\delta' \in X(T)_+$. □

4.2 Applying the lemma repeatedly immediately implies the following proposition.

Proposition . *Assume that $p > h$. Let $\lambda, \mu \in X(T)_+$, both non-zero, and $i \geq 0$. If $\mathrm{Ext}^i_G(V(\lambda)^{(r)}, H^0(\mu)) \neq 0$, then there exists a sequence of non-zero weights $\mu = \gamma_0, \gamma_1, \ldots, \gamma_{r-1}, \gamma_r = \lambda \in X(T)_+$ and nonnegative integers $l_1, l_2, \ldots, l_r$ such that*

(a) $i = \sum_{j=1}^r l_j$,

(b) $\mathrm{Ext}^{l_j}_G(V(\gamma_j)^{(1)}, H^0(\gamma_{j-1})) \neq 0$, *for* $1 \leq j \leq r$ *and*

(c) $\gamma_j = p\delta_j + u_j \cdot 0$ *with* $u_j \in W$ *and nonzero* $\delta_j \in X(T)_+$, *for* $1 \leq j \leq r-1$.

4.3 The next step in our analysis is to obtain tighter control over a lower bound on i as in Proposition 4.2 in the case that $\lambda = \mu$.

Proposition . *Assume that $p > h$. Let $0 \neq \lambda \in X(T)_+$ and $i \geq 0$. If $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$, then there exists a sequence of non-zero weights $\lambda = \gamma_0, \gamma_1, \ldots, \gamma_{r-1}, \gamma_r = \lambda \in X(T)_+$ such that $\gamma_j = p\delta_j + u_j \cdot 0$ for some $u_j \in W$ and nonzero $\delta_j \in X(T)_+$. Furthermore,*

$$i \geq \left(\sum_{j=1}^{r} (p-1)\langle \delta_j, \tilde{\alpha}^\vee \rangle \right) - r. \tag{4.3.1}$$

Equality requires that $p\delta_j - \delta_{j-1} + u_j \cdot 0 = ((l_j - l(u_{j-1}))/2)\tilde{\alpha}$ and that $\langle -u_j \cdot 0, \tilde{\alpha}^\vee \rangle = \ell(u_j) + 1$ for all $1 \leq j \leq r$, where l_j is as in Proposition 4.2.

Proof. The first part is simply a partial restatement of Proposition 4.2 with $\lambda = \mu$. Specifically, there exists a sequence of non-zero dominant weights v$\lambda = \gamma_0, \gamma_1, ..., \gamma_{r-1}, \gamma_r = \lambda$ with $\gamma_j = p\delta_j + u_j \cdot 0$ and corresponding nonnegative integers l_j with $i = \sum_{j=1}^r l_j$ and $\mathrm{Ext}^{l_j}_G(V(\gamma_j)^{(1)}, H^0(\gamma_{j-1})) \neq 0$.

For (4.3.1), we use Proposition 3.4(b) to obtain the inequalities

$$p\langle \delta_j, \tilde{\alpha}^\vee \rangle - \langle \delta_{j-1}, \tilde{\alpha}^\vee \rangle + \ell(u_{j-1}) - \ell(u_j) - 1 \leq l_j \text{ for } 1 \leq j \leq r,$$

with equality only if $p\delta_j - \delta_{j-1} + u_j \cdot 0 = ((l_j - l(u_{j-1}))/2)\tilde{\alpha}$ and $\langle -u_j \cdot 0, \tilde{\alpha}^\vee \rangle = \ell(u_j) + 1$ for all $1 \leq j \leq r$. Note that $\delta_0 = \delta_r$ and $u_0 = u_r$. Summing over j yields

$$\left(\sum_{j=1}^{r} (p-1)\langle \delta_j, \tilde{\alpha}^\vee \rangle \right) - r \leq \sum_{j=1}^{r} l_j = i.$$

□

4.4 For $p > h$ we can now present general vanishing ranges which address (1.1.1).

Theorem . *Assume that $p > h$. Then*

(a) $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) = 0$ *for* $0 < i < r(p-2)$ *and* $\lambda \in X(T)_+$;
(b) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < r(p-2)$.

Proof. Part (a) implies part (b) via Corollary 2.6(B). Suppose that

$$\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$$

for some $0 < i$ and $\lambda \in X(T)_+$. Clearly $\lambda \neq 0$, so we may apply Proposition 4.3. Since $\gamma_j \neq 0$, $1 \leq \langle \delta_j, \tilde{\alpha}^\vee \rangle$. Proposition 4.3 then gives $i \geq r(p-1) - r = r(p-2)$ as claimed. □

Observe that this vanishing range is generally larger than the one obtained in [H]. Precisely, the ranges obtained in [H] are of the form $0 < i < m$ where m depends on the root system. Except in certain type A_n cases, $m \leq r(p-1)/2$.

In the remainder of the paper, we further investigate this question to determine sharp bounds for root systems of type C_n (for all r; see Theorem 5.4) and A_n (for $r = 1$ and generically for all r; see Theorems 6.13, 6.14). In type C_n, the above bounds are in fact sharp.

Remark . Note that the assumption $\mathrm{Ext}^i_G(V(\lambda)^{(r)}, H^0(\mu)) \neq 0$ in Proposition 4.2 can be replaced by $\mathrm{Ext}^k_{G/G_1}(V(\lambda)^{(r)}, \mathrm{H}^l(G_1, H^0(\mu))) \neq 0$, where $k + l = i$. In that case one arrives at the same conclusions with $l_1 = l$. Now the arguments used to prove Proposition 4.3 and Theorem 4.4 can be used to show that $\mathrm{Ext}^k_{G/G_1}(V(\lambda)^{(r)}, \mathrm{H}^l(G_1, H^0(\lambda)) = 0$ for all $k + l < r(p-2)$.

5 Type C_n, $n \geq 1$

Assume throughout this section that Φ is of type C_n, $n \geq 1$, and $p > h = 2n$.

5.1 Realization for $r = 1$**.** We determine the least $i > 0$ such that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$. From Theorem 4.4, we know that $i \geq p-2$. Let $\tilde{\alpha} = 2\omega_1$ denote the longest positive root. We next construct a weight λ with $\mathrm{H}^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$. Let $w = s_{\tilde{\alpha}} = s_1 s_2 \dots s_{n-1} s_n s_{n-1} \dots s_2 s_1 \in W$. Then $-w \cdot 0 = n\tilde{\alpha} = 2n\omega_1$. Furthermore, when expressed as a sum of distinct positive roots, $-w \cdot 0$ consists of precisely all positive roots which contain an α_1. Set $\lambda = p\omega_1 + w \cdot 0 = p\omega_1 - 2n\omega_1 = (p-2n)\omega_1$. Then

$$\lambda - \omega_1 = (p - 1 - 2n)\omega_1 = \left(\frac{p-1}{2} - n\right)\tilde{\alpha}$$

is a highest weight of $S^j(\mathfrak{u}^*)$ where $j = \frac{p-1}{2} - n$.

We will apply Proposition 3.2 to compute $\dim \mathrm{H}^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)})$. Specifically, we will show that for $u \in W$

$$P_{\frac{p-1}{2}-n}(u \cdot \lambda - \omega_1) = \begin{cases} 1 \text{ if } u = 1 \\ 0 \text{ else.} \end{cases}$$

We will work with the ϵ-basis of $X(T)$. Rewrite $u \cdot \lambda - \omega_1 = ((p-2n)u - 1)\epsilon_1 + u \cdot 0$ as $\sum_i c_i \epsilon_i$. In order for this expression to be a sum of positive roots, the coefficient c_1 has to be nonnegative. This forces $u(\epsilon_1) = \epsilon_1$. Then $u \cdot 0$ is of the form $-\sum_i d_i \alpha_i$ with $d_1 = 0$. This implies that $u \cdot \lambda - \omega_1 = (\frac{p-1}{2} - n)\tilde{\alpha} - \sum_i d_i \alpha_i$. Such an expression contains $p - 1 - 2n$

copies of α_1. Since $\tilde{\alpha}$ is the only positive root containing $2\alpha_1$, the above expression can be written as a sum of $\frac{p-1}{2} - n$ positive roots if and only if $u = 1$. From Proposition 3.2, one concludes that $\mathrm{H}^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \cong k$.

5.2 More Vanishing in Degree $p-2$. To get a precise vanishing range, we need to consider whether there are any other weights with non-zero cohomology in degree $p-2$. Let $\lambda = p\mu + w \cdot 0 \in X(T)_+$ with $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(-1)}) \neq 0$ for some $i > 0$. Consider the maximal short root $\alpha_0 = \omega_2$. By Corollary 3.5(a) with $\sigma = \alpha_0$,

$$i \geq (p-1)\langle \mu, \alpha_0^\vee \rangle + \ell(w) + \langle w \cdot 0, \alpha_0^\vee \rangle. \tag{5.2.1}$$

There are three positive roots β with $\langle \beta, \alpha_0^\vee \rangle = 2$ (unless $n = 1$, in which case there is only one and $\alpha_0 = 2\omega_1$). Since $-w \cdot 0$ can be expressed uniquely as a sum of $\ell(w)$ distinct positive roots, we can say $\langle -w \cdot 0, \alpha_0^\vee \rangle \leq \ell(w) + 3$. Hence, (5.2.1) can be rewritten as

$$i \geq (p-1)\langle \mu, \alpha_0^\vee \rangle - 3. \tag{5.2.2}$$

In Type C_n, $\langle \omega_1, \alpha_0^\vee \rangle = 1$. But, for $2 \leq j \leq n$, $\langle \omega_j, \alpha_0^\vee \rangle = 2$. Hence, for $\mu \in X(T)_+$, if $\mu \neq 0, \omega_1$, then $\langle \mu, \alpha_0^\vee \rangle \geq 2$. If $\langle \mu, \alpha_0^\vee \rangle \geq 2$, then (5.2.2) becomes

$$i \geq 2(p-1) - 3 = 2p - 5 > p - 2$$

since $p \geq 5$ ($n \geq 2$). Therefore, the only candidates for a non-zero cohomology group in degree $p-2$ are with $\lambda = p\omega_1 + w \cdot 0$ for some $w \in W$. This makes sense because the weight constructed in Section 5.1 is of this form.

5.3 A Sharp Bound for $r = 1$. Suppose $\lambda = p\omega_1 + w \cdot 0 \in X(T)_+$ and

$$\mathrm{H}^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0.$$

Proposition 3.4 part (b) implies that $\lambda - \omega_1 = (p-1)\omega_1 + w \cdot 0 = ((p-1)/2)\tilde{\alpha} + w \cdot 0 = ((p - 2 - \ell(w))/2)\tilde{\alpha}$ and $\langle w \cdot 0, \tilde{\alpha}^\vee \rangle = -(\ell(w) + 1)$. This forces $w \cdot 0 = -((l(w) + 1)/2)\tilde{\alpha}$. The only possible choices for w satisfying this last equation are $w = 0$ and $w = s_{\tilde{\alpha}}$. Now $\langle w \cdot 0, \tilde{\alpha}^\vee \rangle = -(\ell(w) + 1)$ forces $w = s_{\tilde{\alpha}}$ and $w \cdot 0 = -n\tilde{\alpha}$. Hence, $\lambda = (p - 2n)\omega_1$, the weight given in Section 5.1. So the λ exhibited there is the only dominant weight with $\mathrm{H}^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$.

Note that $\lambda = (p-2n)\omega_1$ is contained in the lowest alcove. There are no smaller dominant weights that are linked to λ (so the condition in Theorem 2.8(A) involving H^{m+1} is vacuous). Consequently, Theorem 2.8(A) and the above discussion now yields:

Theorem . *Suppose Φ is of type C_n with $p > 2n$. Then*
(a) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < p - 2$;
(b) $\mathrm{H}^{p-2}(G(\mathbb{F}_p), k) \cong k$.

5.4 A Sharp Bound for all r. In this section we will address (1.1.1) and (1.1.2) in general for $\mathrm{H}^\bullet(G(\mathbb{F}_q), k)$ when Φ is of type C_n.

Lemma . *Assume $p > 2n$. Let $\lambda = (p - 2n)\omega_1$. Then* $\mathrm{Ext}_G^{r(p-2)}(V(\lambda)^{(r)}, H^0(\lambda)) \cong k$.

Proof. We use induction on r. If $r = 1$ the assertion follows from Section 5.1. Next we make use of the LHS spectral sequence

$$E_2^{k,l} = \mathrm{Ext}^k_{G/G_1}(V(\lambda)^{(r)}, \mathrm{H}^l(G_1, H^0(\lambda)) \Rightarrow \mathrm{Ext}^{k+l}_G(V(\lambda)^{(r)}, H^0(\lambda)).$$

From now on we assume that $E_2^{k,l} \neq 0$. We apply Lemma 4.1 and Remark 4.4 to conclude that there exists a dominant weight $\gamma = p\delta + u\cdot 0$ with $\mathrm{Hom}_G(V(\gamma), \mathrm{H}^l(G_1, H^0(\lambda))^{(-1)}) \neq 0$ and $\mathrm{Ext}^k_G(V(\lambda)^{(r-1)}, H^0(\gamma)) \neq 0$. Furthermore, by Propositions 4.2 and 4.3 there exists a sequence of non-zero weights $\lambda = \gamma_0, \gamma_1, \dots, \gamma_{r-1}, \gamma_r = \lambda \in X(T)_+$ with $\gamma_1 = \gamma$ such that $\gamma_j = p\delta_j + u_j \cdot 0$ for some $u_j \in W$ and nonzero $\delta_j \in X(T)_+$. In addition,

$$k + l \geq \left(\sum_{j=1}^{r} (p-1)\langle \delta_j, \tilde{\alpha}^\vee\rangle\right) - r. \tag{5.4.1}$$

Equality requires that $p\delta_j - \delta_{j-1} + u_j\cdot 0 = ((l_j - l(u_{j-1}))/2)\tilde{\alpha}$ and that $\langle -u_j\cdot 0, \tilde{\alpha}^\vee\rangle = \ell(u_j)+1$ for all $1 \leq j \leq r$, where l_j is as in Proposition 4.2. It follows immediately that $E_2^{k,l} = 0$ whenever $k + l < r(p-2)$.

Looking at $\mathrm{Hom}_G(V(\gamma), \mathrm{H}^l(G_1, H^0(\lambda))^{(-1)}) \cong \mathrm{Hom}_B(V(\gamma), S^{\frac{l-2n+1}{2}}(\mathfrak{u}^*) \otimes \omega_1))$ one concludes that for $l \leq p-2$, all weights μ appearing in $S^{\frac{l-2n+1}{2}}(\mathfrak{u}^*)\otimes\omega_1$ satisfy $\langle \mu+\rho, \alpha_0^\vee\rangle < p$. Hence, $\mathrm{H}^l(G_1, H^0(\lambda))^{(-1)}$ is completely reducible for $l \leq p-2$. From Sections 5.1 and 5.2 one concludes that $H^0(\lambda)$ appears as a summand if and only if $l = p-2$. Clearly the trivial module does not appear as a summand of $\mathrm{H}^l(G_1, H^0(\lambda))^{(-1)}$. But these are the only composition factors of $\mathrm{H}^l(G_1, H^0(\lambda))^{(-1)}$ that could be linked to the weight $p^{r-1}\lambda$. The linkage principle now forces $l \geq p-2$. Moreover, if $l = p-2$, the only possible choice for γ is that $\gamma = \lambda$ and hence $E_2^{k,p-2} \cong \mathrm{Ext}^k_G(V(\lambda)^{(r-1)}, H^0(\lambda))$.

If $k + l = r(p-2)$ then (5.4.1) becomes an equality. This forces $\gamma_1 = \gamma = p\delta + u\cdot 0 = ((l - l(u))/2)\tilde{\alpha} + \omega_1 = (l - 2n + 2)\omega_1$, $\delta_1 = \delta = \omega_1$, and $\langle -u\cdot 0, \tilde{\alpha}^\vee\rangle = \ell(u) + 1$. Using a similar argument to the one in Section 5.3 one concludes that $\gamma = p\omega_1 + s_{\tilde{\alpha}}\cdot 0 = \lambda$, which forces $l = p-2$ and $k = (r-1)(p-2)$.

To summarize, we have shown that

$$E_2^{k,l} \cong \begin{cases} 0 & \text{if } k+l < r(p-2) \\ 0 & \text{if } l < p-2 \\ 0 & \text{if } k+l = r(p-2) \text{ and } l \neq p-2 \\ \mathrm{Ext}^k_G(V(\lambda)^{(r-1)}, H^0(\lambda)) & \text{if } k+l = r(p-2) \text{ and } l = p-2. \end{cases}$$

Therefore, the $((r-1)(p-2), p-2)$-term of the E_2-page transgresses to the E_∞-page and produces an isomorphism $\mathrm{Ext}^{r(p-2)}_G(V(\lambda)^{(r)}, H^0(\lambda)) \cong \mathrm{Ext}^{(r-1)(p-2)}_G(V(\lambda)^{(r-1)}, H^0(\lambda))$, and the claim follows by induction. □

By applying Theorem 4.4, the fact that λ is the smallest weight in its linkage class, and Theorem 2.8(B) one obtains the following theorem.

Theorem . *Suppose Φ is of type C_n with $p > 2n$. Then*

(a) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < r(p-2)$;
(b) $\mathrm{H}^{r(p-2)}(G(\mathbb{F}_q), k) \cong k$.

6 Type A_n, $n \geq 2$

Assume throughout this section that Φ is of type A_n, $n \geq 2$, and that $p > h = n+1$. Note that type A_1 is equivalent to type C_1 which was covered in Section 5.

6.1 An Upper Bound for $r = 1$. We first construct a weight λ with

$$\mathrm{H}^{2p-3}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0.$$

Set $w := s_{\tilde{\alpha}} = s_1 s_2 \dots s_{n-1} s_n s_{n-1} \dots s_2 s_1 \in W$, where s_i is again the reflection corresponding to the ith simple root α_i. Then $\ell(w) = 2n-1$ and $-w \cdot 0 = n\tilde{\alpha}$. When decomposed uniquely into a sum of distinct positive roots, $-w \cdot 0$ consists of precisely all positive roots which contain either an α_1 or an α_n (or both). Set $\lambda := p\tilde{\alpha} + w \cdot 0 = (p-n)\tilde{\alpha}$ and $\mu := \tilde{\alpha}$. Then $\lambda - \mu = (p-n-1)\tilde{\alpha}$ is a weight of $S^{p-n-1}(\mathfrak{u}^*)$. Indeed, it is the highest weight corresponding to taking $(p-n-1)$-copies of $\phi_{\tilde{\alpha}} \in \mathfrak{u}^*$ (the root vector corresponding to $\tilde{\alpha}$). Similar to the argument in Section 5.1, we will show that

$$P_{p-n-1}(u \cdot \lambda - \mu) = \begin{cases} 1 \text{ if } u = 1 \\ 0 \text{ else.} \end{cases}$$

We will work with the ϵ-basis of $X(T)$. Rewrite $u \cdot \lambda - \mu = ((p-n)u - 1)(\epsilon_1 - \epsilon_{n+1})) + u \cdot 0$ as $\sum_i c_i \epsilon_i$. In order for this expression to be a sum of positive roots, the coefficient c_1 has to be nonnegative and c_{n+1}has to be less than or equal to zero. This forces $u(\epsilon_1) = \epsilon_1$ and $u(\epsilon_{n+1}) = \epsilon_{n+1}$. This forces now $u \cdot 0$ to be of the form $-\sum_i d_i \alpha_i$ with $d_1 = d_{n+1} = 0$. This implies that $u \cdot \lambda - \mu = (p-n-1)\tilde{\alpha} - \sum_i d_i \alpha_i$ can be written as a sum of $p-n-1$ positive roots if and only if $u = 1$.

Proposition . *Suppose Φ is of type A_n with $n \geq 2$ and $p \geq n+2$. Let $\lambda = (p-n)\tilde{\alpha} = (p-n)\omega_1 + (p-n)\omega_n$. Then*

(a) $\mathrm{H}^{2p-3}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \cong k$;
(b) $\mathrm{H}^{2p-3}(G(\mathbb{F}_p), k) \neq 0$.

Proof. Part (a) follows from our analysis above and Proposition 3.2.

For part (b) suppose that $0 \neq \mu \in X(T)_+$ is linked to $(p-n)\tilde{\alpha}$ and $\mathrm{H}^i(G, H^0(\mu) \otimes H^0(\mu^*)^{(1)}) \neq 0$ for some $i > 0$. As noted in Section 3.2, we necessarily have $\mu = p\delta + w \cdot 0$ for some $0 \neq \delta \in X(T)_+$ and $w \in W$. Observe that, since μ lies in the root lattice, δ also lies in the root lattice. Therefore, $\langle \delta, \tilde{\alpha}^\vee \rangle \geq 2$. From Corollary 3.5(b) we get $i \geq 2(p-1) - 1 = 2p-3$.

Note also that $(p-n)\tilde{\alpha}$ lies in the second fundamental p-alcove. Indeed, it is the reflection of the zero weight across the upper wall. So the only weight μ of the form $p\tilde{\alpha} + w \cdot 0$ with $\mu < \lambda$ would be the zero weight. And we know that $\mathrm{H}^i(G, k) = 0$ for all $i > 0$. Therefore, we can apply Theorem 2.8(B) to deduce the result.

□

6.2 An Upper Bound for $r > 1$. The following result indentifies a non-zero cohomology class in degree $r(2p-3)$.

Lemma . *Assume $n \geq 2$ and $p > n+2$. Let $\lambda = (p-n)\tilde{\alpha} = (p-n)(\omega_1 + \omega_n)$. Then $\operatorname{Ext}_G^{r(2p-3)}(V(\lambda)^{(r)}, H^0(\lambda)) \cong k$.*

Proof. We use induction on r. If $r = 1$ the assertion follows from Section 6.1. The following argument follows closely the argument in Section 5.4. Again we make use of the LHS spectral sequence

$$E_2^{k,l} = \operatorname{Ext}^k_{G/G_1}(V(\lambda)^{(r)}, \mathrm{H}^l(G_1, H^0(\lambda))) \Rightarrow \operatorname{Ext}_G^{k+l}(V(\lambda)^{(r)}, H^0(\lambda)),$$

and assume that $E_2^{k,l} \neq 0$. We apply Lemma 4.1 and Remark 4.4 to conclude that there exists a dominant weight $\gamma = p\delta + u \cdot 0$ with $\operatorname{Hom}_G(V(\gamma), \mathrm{H}^l(G_1, H^0(\lambda))^{(-1)}) \neq 0$ and $\operatorname{Ext}^k_G(V(\lambda)^{(r-1)}, H^0(\gamma)) \neq 0$. Furthermore, by Propositions 4.2 and 4.3 there exists a sequence of non-zero weights $\lambda = \gamma_0, \gamma_1, \ldots, \gamma_{r-1}, \gamma_r = \lambda \in X(T)_+$ with $\gamma_1 = \gamma$ such that $\gamma_j = p\delta_j + u_j \cdot 0$ for some $u_j \in W$ and nonzero $\delta_j \in X(T)_+$. Note that the linkage principle forces all δ_j to be in the root lattice. Since none of the fundamental weights are contained in the root lattice, $\langle \delta_j, \tilde{\alpha}^\vee \rangle \geq 2$. From (4.3.1), we get

$$k + l \geq \left(\sum_{j=1}^{r} (p-1)\langle \delta_j, \tilde{\alpha}^\vee \rangle \right) - r \geq r(2p-3). \tag{6.2.1}$$

For the first inequality in (6.2.1) to be an equality, from Proposition 4.3, we must have $p\delta_j - \delta_{j-1} + u_j \cdot 0 = ((l_j - l(u_{j-1}))/2)\tilde{\alpha}$ (where l_j is as in Proposition 4.2) and $\langle -u_j \cdot 0, \tilde{\alpha}^\vee \rangle = \ell(u_j) + 1$ for all $1 \leq j \leq r$. Further, for the second inequality to be an equality, clearly we must have $\langle \delta_j, \tilde{\alpha}^\vee \rangle = 2$. It follows immediately that $E_2^{k,l} = 0$ whenever $k + l < r(2p-3)$.

Looking at $\operatorname{Hom}_G(V(\gamma), \mathrm{H}^l(G_1, H^0(\lambda))^{(-1)}) \cong \operatorname{Hom}_B(V(\gamma), S^{\frac{l-2n+1}{2}}(\mathfrak{u}^*) \otimes \tilde{\alpha}))$ one concludes that for $l \leq 2p-3$ the only possible weights γ of the form $p\delta + u \cdot 0$ with δ in the root lattice that make the above expression non-zero are λ and zero. Clearly the trivial module does not appear as a section in a good filtration of $\mathrm{H}^l(G_1, H^0(\lambda))^{(-1)}$ while $H^0(\lambda)$ appears only once. Namely, in the case $l = 2p-3$. The latter follows from the discussion in Section 6.1.

The linkage principle now forces $l \geq 2p-3$. Moreover, if $l = 2p-3$, the only possible choice is $\gamma = \lambda$, and hence $E_2^{k,2p-3} \cong \operatorname{Ext}^k_G(V(\lambda)^{(r-1)}, H^0(\lambda))$.

If $k + l = r(2p-3)$, then (6.2.1) becomes an equality. This forces $\gamma_1 = \gamma = p\delta + u \cdot 0 = ((l - l(u))/2)\tilde{\alpha} + \tilde{\alpha}$ and $\delta_1 = \delta = \tilde{\alpha}$. The only elements u of the Weyl group with $u \cdot 0$ being a multiple of $\tilde{\alpha}$ are the identity and $s_{\tilde{\alpha}}$. Now $\langle -u \cdot 0, \tilde{\alpha}^\vee \rangle = \ell(u) + 1$ forces $\gamma = p\omega_1 + s_{\tilde{\alpha}} \cdot 0 = \lambda$, which forces $l = 2p-3$ and $k = (r-1)(2p-3)$.

As in 5.4 it follows that

$$E_2^{k,l} \cong \begin{cases} 0 & \text{if } k + l < r(2p-3) \\ 0 & \text{if } l < 2p-3 \\ 0 & \text{if } k + l = r(2p-3) \text{ and } l \neq 2p-3 \\ \operatorname{Ext}^k_G(V(\lambda)^{(r-1)}, H^0(\lambda)) & \text{if } k + l = r(2p-3) \text{ and } l = 2p-3. \end{cases}$$

Therefore, the $((r-1)(p-2), p-2)$-term of the E_2-page transgresses to the E_∞-page and produces an isomorphism $\operatorname{Ext}_G^{r(p-2)}(V(\lambda)^{(r)}, H^0(\lambda)) \cong \operatorname{Ext}_G^{(r-1)(p-2)}(V(\lambda)^{(r-1)}, H^0(\lambda))$, and the claim follows by induction. $\square$

Remark . We have actually shown a stronger statement. Namely, for any dominant weight λ of the form $p\delta + u \cdot 0$ with δ in the root lattice, one has

$$\operatorname{Ext}^i_G(V(\lambda)^{(r)}, H^0(\lambda)) \cong \begin{cases} 0 & \text{if } i < 2p-3, \\ 0 & \text{if } i = r(2p-3) \text{ and } \lambda \neq (p-n)(\omega_1+\omega_n), \\ k & \text{if } i = r(2p-3) \text{ and } \lambda = (p-n)(\omega_1+\omega_n). \end{cases}$$

From Theorem 2.8(B) one concludes the following.

Corollary . *Suppose Φ is of type A_n with $n \geq 2$ and $p \geq n+2$. Then $\mathrm{H}^{r(2p-3)}(G(\mathbb{F}_q), k) \neq 0$.*

Corollary 6.2 and Theorem 4.4 imply that the least positive i with $\mathrm{H}^i(G(\mathbb{F}_q), k) \neq 0$ satisfies $r(p-2) \leq i \leq r(2p-3)$. In the following sections, we identify precisely the value of i. The answer will depend on the relationship between p and n.

6.3 Counting Simple Roots. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the simple roots and $\omega_1, \omega_2, ..., \omega_n$ the corresponding fundamental weights. Any weight γ can be written in the form $\gamma = \sum_{j=1}^n c_j\alpha_j$ with $c_j \in \mathbb{Q}$. We define $M_j(\gamma) := c_j$, $M(\gamma) := \max\{c_j\}$, and $m(\gamma) := \max\{j \mid c_j = M(\gamma)\}$. In addition we set $N_j := j(n+1-j)$. We make the following

Observation . For $1 \leq j \leq n$ and $w \in W$,

(a) $\omega_j = \frac{1}{n+1}($ sum of all positive roots which contain $\alpha_j)$,
(b) N_j is the number of positive roots in Φ which contain α_j,
(c) $m(\omega_j) = j$,
(d) $M_{m(\omega_j)}(\omega_j) = M_j(\omega_j) = M(\omega_j) = \frac{N_j}{n+1}$,
(e) $M_j(2\rho) = N_j$,
(f) $M_j(2\rho) = M_j(-w_0 \cdot 0) \geq M_j(-w \cdot 0)$.

Suppose we have dominant weights λ, μ with $\lambda = p\delta_2 + w_2 \cdot 0$, $\mu = p\delta_1 + w_1 \cdot 0$ and $\mathrm{H}^i(G, H^0(\mu) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ for some $i > 0$. As in the proof of Proposition 3.4, $\lambda - \delta_1$ must be a weight of $S^{\frac{i-\ell(w_1)}{2}}(\mathfrak{u}^*)$. Hence $M_{m(\delta_2)}(\lambda - \delta_1) \leq \frac{i-\ell(w_1)}{2}$. Using $\lambda = p\delta_2 + w_2 \cdot 0$, it follows that

$$2pM_{m(\delta_2)}(\delta_2) + 2M_{m(\delta_2)}(w_2 \cdot 0) - 2M_{m(\delta_2)}(\delta_1) + l(w_1) \leq i. \tag{6.3.1}$$

Note that $M_{m(\delta_2)}(\delta_1) \leq M(\delta_1) = M_{m(\delta_1)}(\delta_1)$ and that $-M_{m(\delta_1)}(w_1 \cdot 0) \leq l(w_1)$ (from Observation 2.2(B)). One obtains

$$2pM_{m(\delta_2)}(\delta_2) + 2M_{m(\delta_2)}(w_2 \cdot 0) - 2M_{m(\delta_1)}(\delta_1) - M_{m(\delta_1)}(w_1 \cdot 0) \leq i. \tag{6.3.2}$$

Next assume that $\mu = \lambda$. Then (6.3.2) results in $2(p-1)M_{m(\delta_1)}(\delta_1) + M_{m(\delta_1)}(w_1 \cdot 0) \leq i$. Using Observation (f) gives

$$2(p-1)M_{m(\delta_1)}(\delta_1) + M_{m(\delta_1)}(-2\rho) \leq i. \tag{6.3.3}$$

Suppose now that $\delta_1 = \omega_j$ for $1 \leq j \leq n$. From Observations (d) and (e), (6.3.3) becomes

$$2\left(\frac{p-1}{n+1}\right)N_j - N_j \leq i. \tag{6.3.4}$$

However, we can say more than this. Suppose that in fact $M_j(-w_1 \cdot 0) = N_j$. Then, when expressed as a sum of distinct positive roots, $-w_1 \cdot 0$ contains all N_j roots containing α_j and possibly some other positive roots. In other words, $-w_1 \cdot 0 = (n+1)\omega_j + \sigma$ where σ is a sum of distinct positive roots not containing α_j. Then

$$\lambda = p\omega_j + w_1 \cdot 0 = p\omega_j - (n+1)\omega_j - \sigma = (p-n-1)\omega_j - \sigma \leq (p-n-1)\omega_j.$$

Hence, the only way λ can be dominant is if $\sigma = 0$. In other words, $\lambda = (p-n-1)\omega_j$ and we have shown the following.

Proposition . *Suppose that Φ is of type A_n with $n \geq 2$ and $\lambda = p\omega_j + w \cdot 0 \in X(T)_+$ for $w \in W$ is a weight of $S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*) \otimes \omega_j$. Then*

$$i \geq \left[2\left(\frac{p-1}{n+1}\right) - 1\right] j(n+1-j)$$

with equality possible if and only if $\lambda = (p-n-1)\omega_j$.

Remark . The assumption that $\lambda = p\omega_j + w \cdot 0$ is a weight of $S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*) \otimes \omega_j$ places restrictions on p (and n). Indeed, $\lambda - \omega_j = (p-1)\omega_j + w \cdot 0$ must lie in the root lattice. But $w \cdot 0$ lies in the root lattice. Therefore, $(p-1)\omega_j$ must also lie in the root lattice. However, for $1 \leq j \leq (n+1)/2$, to have $(p-1)\omega_j$ or symmetrically $(p-1)\omega_{n+1-j}$ in the root lattice, we must have that $(n+1)$ divides $(p-1)j$.

Given the constraints noted in the remark, it is useful to rewrite the right hand side of the proposition as

$$\left[2\left(\frac{p-1}{n+1}\right) - 1\right] j(n+1-j) = \big[2(p-1) - (n+1)\big] j \left(1 - \frac{j}{n+1}\right). \tag{6.3.5}$$

6.4 Larger Weights. In this section, we will see that the only non-zero non-fundamental dominant weight λ which can have $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ for $0 \leq i \leq 2p-3$ is the weight $\lambda = (p-n)\tilde{\alpha}$ considered in Section 6.1. Indeed, observe that when $p \geq n+2$, $2(p-1) + 2(n-1)\left[\frac{p-1}{n+1} - 1\right] > 2p-3$.

Proposition . *Suppose that Φ is of type A_n with $n \geq 2$ and $p \geq n+2$. Let $\lambda = p\mu + w \cdot 0 \in X(T)_+$ for $w \in W$ with $\langle \mu, \tilde{\alpha} \rangle \geq 2$. If $\lambda \neq (p-n)\tilde{\alpha}$, then $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = 0$ for*

$$0 < i \leq 2(p-1) + 2(n-1)\left[\frac{p-1}{n+1} - 1\right].$$

Proof. Let λ and μ be as given. Assume that $i \neq 0$ and $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$. Using our definition of $m(\mu)$ from Section 6.3 we write $\mu = a\omega_{m(\mu)} + \sigma$, where σ is a sum of fundamental weights other than $\omega_{m(\mu)}$ and a is a positive integer. Note that σ is nonzero if $a = 1$. Set $j = \min\{m(\mu), n+1-m(\mu)\}$. Note that $M_{m(\mu)}(\omega_l) \geq \frac{j}{n+1}$ for any $l \neq j$. We

obtain the following inequality

$$M_{m(\mu)}(\mu) \geq \begin{cases} \frac{2N_j}{n+1} & \text{if } a \geq 2, \\ \\ \frac{N_j+j}{n+1} & \text{else.} \end{cases}$$

Substituting the above into equation (6.3.3) with $\delta_1 = \mu$ yields

$$i \geq \begin{cases} 2\left(\frac{p-1}{n+1}\right)(2N_j) - N_j & \text{if } a \geq 2, \\ \\ 2\left(\frac{p-1}{n+1}\right)(N_j + j) - N_j & \text{else.} \end{cases}$$

As a function of j, N_j is increasing on the interval $(0, (n+1)/2)$. Therefore both of the above expressions are minimal when j is as small as possible. In the first case $j = 1$ is possible. However, since we are assuming that $\mu \neq \omega_1 + \omega_n$, we may assume that $j \geq 2$ for the second case. One obtains

$$\text{(6.4.1)} \qquad i \geq \begin{cases} 2\left(\frac{p-1}{n+1}\right)2n - n & \text{if } a \geq 2, \\ \\ 2\left(\frac{p-1}{n+1}\right)(2(n-1)+2) - 2(n-1) = 2\left(\frac{p-1}{n+1}\right)2n - 2(n-1) & \text{else.} \end{cases}$$

Hence,

$$\begin{aligned} i &\geq 4n\left(\frac{p-1}{n+1}\right) - 2(n-1) \\ &= 4(n+1)\left(\frac{p-1}{n+1}\right) - 4\left(\frac{p-1}{n+1}\right) - 2(n-1) \\ &= 2(p-1) + [2(n+1)-4]\left(\frac{p-1}{n+1}\right) - 2(n-1) \\ &= 2(p-1) + 2(n-1)\left(\frac{p-1}{n+1}\right) - 2(n-1) \\ &= 2(p-1) + 2(n-1)\left[\frac{p-1}{n+1} - 1\right]. \end{aligned}$$

□

To determine sharp vanishing bounds, we need to consider the relationship between p and n. This will be done in the succeeding sections.

6.5 The Case: $\gcd(p-1, n+1) = 1$**.** It follows from Remark 6.3 that under this assumption, the weight $(p-1)\omega_j$ does not lie in the root lattice for any j. Therefore, μ must be the sum of at least two (not necessarily distinct) fundamental dominant weights, and $\langle \mu, \tilde{\alpha}^\vee \rangle \geq 2$. From Corollary 3.5(b), we conclude that $i \geq 2p-3$. Combining this with Proposition 6.1, Proposition 6.4, and Theorem 2.8(A), we obtain these sharp bounds.

Theorem . *Suppose Φ is of type A_n with $n \geq 2$, $p \geq n+2$ and* $\gcd(p-1, n+1) = 1$. *Then*

(a) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-3$*;*

(b) $\mathrm{H}^{2p-3}(G(\mathbb{F}_p), k) \cong k$.

6.6 The Case: $1 < \gcd(p-1, n+1) < n+1$**.** For convenience, set $g := \gcd(p-1, n+1)$. We investigate here the inequality in Proposition 6.3. Note that since $n+1$ does not divide $p-1$, neither $(p-1)\omega_1$ nor $(p-1)\omega_n$ lie in the root lattice. So we restrict attention to ω_j with $1 < j < n$. As such, there is nothing to consider unless $n \geq 3$. Without a loss of generality assume that $j \leq (n+1)/2$. Furthermore, from Remark 6.3, we may assume that $j(p-1)$ is divisible by $(n+1)$.

Consider the function $f(j) = \left[2\left(\frac{p-1}{n+1}\right) - 1\right] j(n+1-j)$, which is a quadratic polynomial in the variable j. For our purposes, we want to minimize $f(j)$. This evidently occurs when j is minimal (for $j \leq (n+1)/2$). So we consider the case that j is minimal such that $j(p-1)$ is divisible by $(n+1)$. This implies that, $n+1 = gj$, where $g = \gcd(n+1, p-1)$. With this subsitution, using (6.3.5), the inequality in Proposition 6.3 may be rewritten as

$$i \geq \big[2(p-1) - (n+1)\big] j \left(1 - \frac{1}{g}\right). \tag{6.6.1}$$

Suppose first that $n+1 = gj$ is odd (and $n \geq 4$). Then, both g and j must be odd. Therefore, $g \geq 3$ and $3 \leq j \leq (n+1)/2$. Hence,

$$j\left(1 - \frac{1}{g}\right) \geq 3\left(1 - \frac{1}{3}\right) = 2.$$

Equation (6.6.1) allows us to conclude that

$$\begin{aligned}
i &\geq 2\big[2(p-1) - (n+1)\big] \\
&= 4(p-1) - 2(n+1) \\
&= 2(p-1) + 2(p-1) - 2(n+1) \\
&= 2(p-1) + 2(p-n-2) \\
&\geq 2(p-1) = 2p-2
\end{aligned}$$

since $p \geq n+2$. So we get a bound on i which is strictly larger than $2p-3$.

Consider now the case that $n+1 = gj$ is even (and $n \geq 3$). Since $p-1$ is even, g is necessarily even. In particular, $g \geq 2$. Since $g \neq n+1$, we also have $j \geq 2$. Suppose first that $j \geq 4$. Then

$$j\left(1 - \frac{1}{g}\right) \geq 4\left(1 - \frac{1}{2}\right) = 2.$$

And so the same argument as in the $n+1$-odd case would show that $i \geq 2p-2$.

Suppose next that $j = 3$ and $g > 2$. Then $g \geq 4$. Then,

$$j\left(1 - \frac{1}{g}\right) \geq 3\left(1 - \frac{1}{4}\right) = \frac{9}{4} > 2.$$

We conclude that $i > 2p-2$.

Suppose next that $j = 3$ and $g = 2$. Then $n + 1 = 6$ and $p - 1 > n + 1 = 6$. So $p \geq 11$ (as p is prime). Here we get

$$\begin{aligned} i &\geq \big[2(p-1) - (n+1)\big] j \left(1 - \frac{1}{g}\right) \\ &= \big[2(p-1) - 6\big] 3 \left(1 - \frac{1}{2}\right) \\ &= 3\big[(p-1) - 3\big] \\ &= 3p - 12 \\ &= 2p - 3 + p - 9 \\ &\geq 2p - 1 \end{aligned}$$

since $p \geq 11$.

Suppose next that $j = 2$, i.e., $n + 1 = 2g$. Since $p - 1 > n + 1$ and g divides $p - 1$, we must have $p - 1 \geq 3g$. Write $p - 1 = (3 + m)g$ for an integer $m \geq 0$. Here we get

$$\begin{aligned} i &\geq \big[2(p-1) - (n+1)\big] j \left(1 - \frac{1}{g}\right) \\ &= \big[2(p-1) - 2g\big] 2 \left(1 - \frac{1}{g}\right) \\ &= 4(p-1) - \frac{4(p-1)}{g} - 4g + 4 \\ &= 2(p-1) + 2(p-1) - \frac{4(p-1)}{g} - 4g + 4 \\ &= 2(p-1) + 2(3+m)g - 4(3+m) - 4g + 4 \\ &= 2(p-1) + 2g - 8 + m(2g-4) \\ &\geq 2(p-1) + 2g - 8 \end{aligned}$$

since $m \geq 0$ and $g \geq 2$. If $g \geq 4$, then we conclude that $i \geq 2p - 2$.

However, if $g = 2$, we can only conclude that $i \geq 2p - 6$. This happens when $n + 1 = gj = 4$ or $n = 3$. Note that for $n = 3$, we either have $g = 2$ or $g = 4$ with the latter case falling into the $n + 1$ divides $p - 1$ category. The case of $n = 3$ will be dealt with specifically in Section 6.11. We summarize our findings in the following proposition.

Proposition . *Suppose Φ is of type A_n with $n \geq 4$. Suppose further that $p > n + 2$ and $1 < \gcd(p-1, n+1) < n + 1$. Let $\lambda = p\omega_j + w \cdot 0 \in X(T)_+$ for $2 \leq j \leq n - 1$ and $w \in W$. Then $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = 0$ for $0 < i \leq 2p - 3$.*

6.7 The Case: $p - 1 = n + 1$. Under this condition, we can explicitly construct a weight λ with $\mathrm{H}^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$. Let $\lambda = p\omega_1 + w \cdot 0$ where $w = s_1 s_2 s_3 \dots s_n$ with s_i denoting the simple reflection corresponding to the ith simple root. Then $-w \cdot 0$ is the sum of all n positive roots containing α_1. In other words, $-w \cdot 0 = (n+1)\omega_1$. So

$\lambda = p\omega_1 + w \cdot 0 = p\omega_1 - (n+1)\omega_1 = \omega_1$. Since $\ell(w) = n = p-2$, by Proposition 3.2,

$$\dim \mathrm{H}^{p-2}(G, H^0(\omega_1) \otimes H^0(\omega_1^*)^{(1)}) = \sum_{u \in W} (-1)^{\ell(u)} P_0(u \cdot \omega_1 - \omega_1) = 1.$$

Note that $\omega_1^* = \omega_n$ and one can similarly argue that $\mathrm{H}^{p-2}(G, H^0(\omega_n) \otimes H^0(\omega_n^*)^{(1)}) \cong k$.

Theorem . *Suppose Φ is of type A_n with $n \geq 2$ and $p - 1 = n + 1$. Then*

(a) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < p-2$*;*
(b) $\mathrm{H}^{p-2}(G(\mathbb{F}_p), k) \cong k \oplus k$.

Proof. Part (a) follows from Theorem 4.4. For part (b), from the discussion above, we know that $\mathrm{H}^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \cong k$ for $\lambda = \omega_1$ or $\lambda = \omega_n$. We claim that if $\mathrm{H}^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ for a dominant weight λ, then $\lambda = \omega_1$ or ω_n. To see this, write $\lambda = p\mu + w \cdot 0$ for some $\mu \in X(T)_+$ and $w \in W$. By Corollary 3.5(b), μ must be a fundamental dominant weight. Apply Proposition 6.3 with $p - 1 = n + 1$. The proposition gives that, for $1 < j < n$,

$$i \geq j(n+1-j) = j(p-1-j) \geq 2(p-1-2) = 2p-6 = (p-2) + (p-4).$$

For $n \geq 3$, $p = n+2 \geq 5$ and so this gives $i > p-2$. Hence, $\mu = \omega_1$ or ω_n. In other words $\lambda = p\omega_1 + w \cdot 0$ or $\lambda = p\omega_n + w \cdot 0$, respectively. From the proof of Corollary 3.5, we observe that in order to have $\mathrm{H}^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$, when $-w \cdot 0$ is expressed as a sum of distinct positive roots, one of those roots must be $\tilde{\alpha} = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. From Observation 2.2(B), it follows that $\ell(w) \geq n = p-2$. Applying Proposition 3.2, we see that

$$\dim \mathrm{H}^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = \sum_{u \in W} (-1)^{\ell(u)} P_0(u \cdot \lambda - \mu).$$

One has non-zero cohomology only if $u \cdot \lambda - \mu = 0$ which can only happen if $\lambda = \mu$, which gives the claim.

Since the only dominant weight less than ω_1 or ω_n is the zero weight, $\mathrm{H}^i(G, k) = 0$ for $i > 0$, and ω_1 and ω_n lie in different linkage classes, the discussion in Section 2.8 gives part (b). □

6.8 The Case: $\gcd(p-1, n+1) = n+1 < p-1$**.** The case $p-1 = n+1$ is excluded since that was dealt with in Section 6.7. Since $n+1$ divides $p-1$, $(p-1)\omega_j$ lies in the root lattice for all $1 \leq j \leq n$, and we need to allow $\lambda = p\omega_j + w \cdot 0$ for all j in our general argument.

Write $p - 1 = d(n+1)$ for an integer $d \geq 2$. We can rewrite the inequality in Proposition 6.3 (see also (6.3.5)) as

$$\begin{aligned}
i &\geq 2j(p-1) - 2j^2\left(\frac{p-1}{n+1}\right) - j(n+1) + j^2 \\
&= 2(p-1) + (2j-2)(p-1) - 2j^2\left(\frac{p-1}{n+1}\right) - j(n+1) + j^2 \\
&= 2(p-1) + \left[(2j-2)(n+1) - 2j^2\right]\left(\frac{p-1}{n+1}\right) - j(n+1) + j^2 \\
&= 2(p-1) + \left[(2j-2)(n+1) - 2j^2\right]d - j(n+1) + j^2.
\end{aligned}$$

For $j = 1$ (or $j = n$), this inequality allows for a value of $i < 2p - 3$. This will be discussed more in the next section. For this section, we focus on the case $2 \leq j \leq n - 1$. By default, we need $n \geq 3$. For such j, the least value of the right hand side above occurs when $j = 2$ (or $j = n - 1$). Substituting $j = 2$, the above inequality becomes

$$i \geq 2(p-1) + (2n-6)d - 2n + 2. \tag{6.8.1}$$

Since $d \geq 2$, (6.8.1) becomes

$$i \geq 2(p-1) + 2n - 10.$$

If $n \geq 5$, then $i \geq 2p - 2$. If $n = 4$ with $d \geq 3$, then (6.8.1) becomes

$$i \geq 2(p-1) + (2*4-6)3 - 2*4 + 2 = 2p - 2.$$

If $n = 4$ and $d = 2$, then $p - 1 = 2(4 + 1) = 10$ or $p = 11$ and we can only say that $i \geq 2p - 6$. This case will be considered in Section 6.12. For $n = 3$, notice that the value of d is irrelevant in (6.8.1). Irrespective of d, we conclude that $i \geq 2p - 6$. This case will be discussed in Section 6.11. We summarize the conclusions of this section in the following.

Proposition . *Suppose Φ is of type A_n with $n \geq 4$. Suppose further that $p > n + 2$ and $\gcd(p-1, n+1) = n+1$. If $n = 4$, assume further that $p \neq 11$. Let $\lambda = p\omega_j + w \cdot 0 \in X(T)_+$ for $2 \leq j \leq n-1$ and $w \in W$. Then $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = 0$ for $0 < i \leq 2p - 3$.*

6.9 The Case: $\gcd(p-1, n+1) = n+1 < p-1$**, continued.** In this section we investigate the case of $\lambda = p\omega_1 + w \cdot 0$ (or symmetrically, $\lambda = p\omega_n + w \cdot 0$).

Lemma . *Suppose that Φ is of type A_n Let $\lambda = p\omega_1 + w \cdot 0 \in X(T)_+$ or $\lambda = p\omega_n + w \cdot 0 \in X(T)_+$ and $\mu = p\omega_j + v \cdot 0 \in X(T)_+$ for $1 \leq j \leq n$ and $v, w \in W$. Then $\mathrm{H}^i(G, H^0(\mu) \otimes H^0(\lambda^*)^{(1)}) = 0$ for $0 < i < (p-1-n)n - (n+1-j)j + \ell(v)$.*

Proof. We give the argument for ω_1. An analogous argument works for ω_n. If $\lambda = p\omega_1 + w \cdot 0$ is dominant, then a direct computation shows that $w = s_k s_{k-1} \ldots s_1$ and $\lambda = (p-k-1)\omega_1 + \omega_{k+1}$, where $0 \leq k \leq n$. Here we are using the conventions $s_0 = 1$ and $\omega_{n+1} = 0$. Moreover, we have the following equations for the formal characters

$$\mathrm{char}(V((p-k)\omega_1) \otimes V(\omega_k)) = \mathrm{char}\, V((p-k)\omega_1 + \omega_k) + \mathrm{char}\, V((p-k-1)\omega_1 + \omega_{k+1}).$$

As discussed earlier, the module $\mathrm{ind}_B^G(S^m(\mathfrak{u}^*) \otimes \omega_j)$ has a good filtration. We conclude that

$$\begin{aligned} &\dim \mathrm{Hom}_G(V(p\omega_1 + w \cdot 0), \mathrm{ind}_B^G(S^m(\mathfrak{u}^*) \otimes \omega_j)) \\ &\leq \dim \mathrm{Hom}_G(V((p-k)\omega_1) \otimes V(\omega_k), \mathrm{ind}_B^G(S^m(\mathfrak{u}^*) \otimes \omega_j)) \\ &= \dim \mathrm{Hom}_G(V((p-k)\omega_1), \mathrm{ind}_B^G(S^m(\mathfrak{u}^*) \otimes \omega_j) \otimes H^0(\omega_{n+1-k})), \end{aligned}$$

where $k = \ell(w)$.

Note that $\langle \omega_j - u(\omega_k), \alpha_l^\vee \rangle \geq -1$ for all $u \in W$ and $1 \leq l \leq n$. It follows from [KLT] that $R^j \mathrm{ind}_B^G(S^m(\mathfrak{u}^*) \otimes (\omega_j + u(\omega_{n+1-k}))) = 0$ for $j > 0$. Using the long exact sequence that one

obtains from a B-filtration of $H^0(\omega_{n+1-k})$ one concludes that

$$\begin{aligned}
&\dim \mathrm{Hom}_G(V(p\omega_1 + w\cdot 0), \mathrm{ind}_B^G(S^m(\mathfrak{u}^*)\otimes\omega_j))\\
&\leq \dim \mathrm{Hom}_G(V((p-k)\omega_1), \mathrm{ind}_B^G(S^m(\mathfrak{u}^*)\otimes\omega_j)\otimes H^0(\omega_{n+1-k}))\\
&= \frac{1}{|\mathrm{Stab}_W(\omega_k)|}\sum_{u\in W}\dim \mathrm{Hom}_G(V((p-k)\omega_1), \mathrm{ind}_B^G(S^m(\mathfrak{u}^*)\otimes\omega_j - u(\omega_k))).
\end{aligned}$$

The Weyl module $V((p-k)\omega_1)$ has one-dimensional weight spaces [J1, II 2.16]. A theorem of Kostant [Hum1, 24.2] implies that

$$\sum_{m\geq 0}\sum_{x\in W}(-1)^{\ell(x)}P_m(x\cdot((p-k)\omega_1) - \omega_j + u(\omega_k)) = 1.$$

According to [AJ, 3.8] and [KLT] we know that

$$\begin{aligned}
&\sum_{x\in W}(-1)^{\ell(x)}P_m(x\cdot((p-k)\omega_1) - \omega_j + u(\omega_k))\\
&= \dim \mathrm{Hom}_G(V((p-k)\omega_1), \mathrm{ind}_B^G(S^m(\mathfrak{u}^*)\otimes\omega_j - u(\omega_k))\geq 0,
\end{aligned}$$

for all $m\geq 0$. Clearly, for $x\neq 1$,

$$\text{height}((p-k)\omega_1 - \omega_j + u(\omega_k)) > \text{height}(x\cdot((p-k)\omega_1) - \omega_j + u(\omega_k)).$$

We conclude that

$$\sum_{x\in W}(-1)^{\ell(x)}P_m(x\cdot((p-k)\omega_1) - \omega_j + u(\omega_k)) = \begin{cases}1 \text{ if } m = \text{ height}((p-k)\omega_1 - \omega_j + u(\omega_k))\\ 0 \text{ else.}\end{cases}$$

We have

$$\begin{aligned}
\text{height}((p-k)\omega_1 - \omega_j + u(\omega_k)) &\geq \text{ height}((p-k)\omega_1 - \omega_j - \omega_{n+1-k})\\
&= \frac{(p-k)n}{2} - \frac{(n+1-j)j}{2} - \frac{(n+1-k)k}{2}.
\end{aligned}$$

Hence,

$$\dim \mathrm{Hom}_G(V(p\omega_1 + w\cdot 0), \mathrm{ind}_B^G(S^m\mathfrak{u}^*\otimes\omega_j)) = 0,$$

if

$$m < \frac{(p-k)n - (n+1-j)j - (n+1-k)k}{2}$$

where $k = \ell(w)$.

Setting $m = \frac{i-\ell(v)}{2}$ and applying Lemma 3.1 and (3.2.1) yields

$$\mathrm{H}^i(G, H^0(p\omega_1 + w\cdot 0)\otimes H^0((p\omega_j + v\cdot 0)^*)^{(1)}) = 0$$

for

$$\frac{i-\ell(v)}{2} < \frac{(p-k)n - (n+1-j)j - (n+1-k)k}{2}.$$

Hence, one obtains vanishing for

$$i < (p-k)n - (n+1-k)k - (n+1-j)j + \ell(v). \tag{6.9.1}$$

Since the global minimum of $(p-k)n-(n+1-k)k$ on the closed interval $[1,n]$ occurs at $k=n$, we get the claimed vanishing for

$$i \leq (p-1-n)n-(n+1-j)j+\ell(v).$$

□

Proposition . *Suppose Φ is of type A_n with $n \geq 3$. Suppose further that $p > n+2$ and $\gcd(p-1,n+1) = n+1$. Let $\lambda = p\omega_1 + w\cdot 0 \in X(T)_+$ or $\lambda = p\omega_n + w \cdot 0$ with $w \in W$. Then $\mathrm{H}^i(G, H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = 0$ for $0 < i \leq 2p-3$.*

Proof. Equation (6.9.1) implies, for the case $\lambda=\mu$, vanishing for $0 < i < (p-k)n-(n+1-k)k-n+\ell(v) = (p-k)n-(n-k)k-n$. Again the global minimum occurs at $k=n$ and one obtains vanishing for $0<i\leq (p-1-n)n$. Observe that $p \geq 2(n+1)+1 = 2n+3$. Now

$$\begin{aligned}(p-1-n)n &= 2p-3+p(n-2)-n(n+1)+3\\ &\geq 2p-3+(2n+3)(n-2)-n^2-n+3\\ &= 2p-3+(n-1)^2-4\\ &\geq 2p-3\end{aligned}$$

with equality if and only if $n=3$ and $p=2n+3$. This case does not occur. □

6.10 The Case: $n=2$. Assume for this subsection that Φ is of type A_2 with $p>3$. From Proposition 6.1 we know that $\mathrm{H}^{2p-3}(G(\mathbb{F}_p),k) \neq 0$. Proposition 6.5 implies that $2p-3$ is indeed the lowest bound unless 3 divides $p-1$. Note that the case $p-1=3$ is not possible for a prime p. If 3 divides $p-1$, then the only possible non-zero cohomology in lower degrees would come from weights of the form $\lambda = p\omega_1 + w\cdot 0$ or the dual case $\lambda = p\omega_2 + w\cdot 0$. It follows from Lemma 6.9 (see also the proof of Proposition 6.9) that $\mathrm{H}^i(G,H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = 0$ for $i < 2p-6$. Moreover, using the arguments of Lemma 6.9 one can show that $\mathrm{H}^i(G,H^0(\lambda)\otimes H^0(\lambda^*)^{(1)}) = 0$ for $i \leq 2p-6$ unless $\lambda = (p-3)\omega_j$. In that case $\sum_{u\in W}(-1)^{\ell(u)}P_{p-4}(u\cdot((p-3)\omega_j)-\omega_j) = 1$ because the height of $(p-4)\omega_j$ is exactly $p-4$. Proposition 3.2 now says that $\mathrm{H}^{2p-6}(G,H^0((p-3)\omega_j)\otimes H^0((p-3)\omega_{(3-j)})^{(1)}) \cong k$, $j=1,2$. Note that ω_1 and ω_2 are in different linkage classes. We conclude the following from Theorem 2.8(A) and the linkage discussion in Section 2.8.

Proposition . *Suppose Φ is of type A_2 and $p>3$.*

(a) *If 3 divides $p-1$, then*
   (i) $\mathrm{H}^i(G(\mathbb{F}_p),k) = 0$ *for* $0<i<2p-6$;
   (ii) $\mathrm{H}^{2p-6}(G(\mathbb{F}_p),k) \cong k\oplus k$.

(b) *If 3 does not divide $p-1$, then $\mathrm{H}^i(G(\mathbb{F}_p),k) = 0$ for $0<i<2p-3$.*

6.11 The Case: $n=3$. Let Φ be of type A_3 with $p>4$. The case $p=5$ is included in Proposition 6.7. For the remainder of this section we assume that $p>5$.

Lemma . *Suppose that Φ is of type A_3 with $p > 4$. Then*

$$\sum_{u\in W}(-1)^{\ell(u)}P_{p-5}(u\cdot((p-4)\omega_2)-\omega_2)=1.$$

Proof. Observe that $2\omega_2=\alpha_1+2\alpha_2+\alpha_3=\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4$. For $u\cdot((p-4)\omega_2)-\omega_2$ to be a sum of positive roots one needs $u(2\omega_2)$ to be a sum of positive roots. This is the case if and only if either $u(2\omega_2)=2\omega_2$ or $u(2\omega_2)=\alpha_1+\alpha_3$. But we can rule out the second case because here $u\cdot((p-4)\omega_2)-\omega_2=((p-5)/2)(\alpha_1+\alpha_3)-\alpha_2+u\cdot 0$. This is clearly not the sum of positive roots. It follows that we only have to consider $u\in \mathrm{Stab}_W(\omega_2)=\{1,s_1,s_3,s_1s_3\}$ and that $u\cdot((p-4)\omega_2)-\omega_2=(p-5)\omega_2+u\cdot 0$. A direct computation now shows that

$$P_{p-5}((p-5)\omega_2+u\cdot 0)=\begin{cases}(p-5)/2 & \text{if } u=1\\ (p-7)/2 & \text{if } u=s_1,s_3,s_1s_3.\end{cases}$$

Hence, $\sum_{u\in W}(-1)^{\ell(u)}P_{p-5}(u\cdot((p-4)\omega_2)-\omega_2)=\sum_{u\in \mathrm{Stab}_W(\omega_2)}(-1)^{\ell(u)}P_{p-5}(u\cdot((p-4)\omega_2)-\omega_2)=1$. □

By Proposition 6.1 we know that $\mathrm{H}^{2p-3}(G(\mathbb{F}_p),k)\neq 0$. Note that $\gcd(p-1,n+1)=\gcd(p-1,4)=2$ or 4. If $\gcd(p-1,4)=2$, then by Remark 6.3, the only possible non-zero cohomology in lower degrees would come from weights of the form $\lambda=p\omega_2+w\cdot 0$. On the other hand, if $\gcd(p-1,4)=4$, then Proposition 6.9 gives the same conclusion.

Suppose $\lambda=p\omega_2+w\cdot 0$. It follows from Proposition 6.3 that $\mathrm{H}^i(G,H^0(\lambda)\otimes H^0(\lambda^*)^{(1)})=0$ for $i<2p-6$ and that $\mathrm{H}^{2p-6}(G,H^0(\lambda)\otimes H^0(\lambda^*)^{(1)})=0$ unless $\lambda=(p-4)\omega_2$. That occurs if $w\cdot 0=-4\omega_2$. By direct computation one finds that $\ell(w)=4$. By Proposition 3.2 and Lemma 6.11 above, we have

$$\begin{aligned}\dim \mathrm{H}^{2p-6}&(G,H^0((p-4)\omega_2)\otimes H^0((p-4)\omega_2^*)^{(1)})\\ &=\sum_{u\in W}(-1)^{\ell(u)}P_{\frac{(2p-6)-4}{2}}(u\cdot((p-4)\omega_2)-\omega_2)\\ &=\sum_{u\in W}(-1)^{\ell(u)}P_{p-5}(u\cdot((p-4)\omega_2)-\omega_2)=1.\end{aligned}$$

Combining this with Theorem 2.8(A) one obtains that $\mathrm{H}^{2p-6}(G(\mathbb{F}_p),k)\cong k$. We summarize our findings for $\Phi=A_3$ below.

Proposition . *Suppose Φ is of type A_3 and $p>4$.*

(a) *If $p=5$, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p),k)=0$ *for* $0<i<p-2$;
  (ii) $\mathrm{H}^{p-2}(G(\mathbb{F}_p),k)\cong k\oplus k$.

(b) *If $p>5$, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p),k)=0$ *for* $0<i<2p-6$;
  (ii) $\mathrm{H}^{2p-6}(G(\mathbb{F}_p),k)\cong k$.

6.12 The Case: $n=4$ and $p=11$. Assume for this section that Φ is of type A_4 with $p=11$. Then $\gcd(p-1,n+1)=\gcd(10,5)=5$ and $2p-3=19$. It follows from Proposition 6.9 and Proposition 6.4 that the only non-zero cohomology in degrees lower than 19 has to

come from weights of the form $\lambda = 11\omega_2 + w \cdot 0 = 6\omega_2$ or $\lambda = 11\omega_3 + w \cdot 0 = 6\omega_3$. In these cases we could potentially achieve non-vanishing in degree 18. The weights ω_2 and ω_3 are dual. We give the argument for ω_2. An analogous argument works for ω_3. Here $\ell(w) = 6$. According to Proposition 3.2, to have $\mathrm{H}^{18}(G, H^0((p-5)\omega_2) \otimes H^0((p-5)\omega_3)^{(1)}) \neq 0$ one needs $\sum_{u \in W} (-1)^{\ell(u)} P_6(u \cdot (6\omega_2) - \omega_2) \neq 0$. The lemma below (Lemma 6.12) rules this case out.

According to Proposition 6.1, $\mathrm{H}^{19}(G(\mathbb{F}_p), k) \neq 0$. Again it follows from Proposition 6.9 and Proposition 6.4 that any cohomology in degree 19 other then the one coming from Proposition 6.1 has to come from weights of the form $\lambda = 11\omega_2 + w \cdot 0$ or $\lambda = 11\omega_3 + w \cdot 0$. Suppose $\lambda = 11\omega_2 + w \cdot 0$. Again, the ω_3 case is analogous. Proposition 3.2 implies that cohomology in odd degrees has to come from weights with corresponding $w \in W$ of odd length. Now equation (6.3.1) shows that this can only happen if $\ell(w) = 5$. Note that here $M_2(-w \cdot 0) = \ell(w) = 5$. The only such weight is $\lambda = 6\omega_2 + \tilde{\alpha}$. By Proposition 3.2 it suffices to show that $\sum_{u \in W} (-1)^{\ell(u)} P_7(u \cdot (6\omega_2 + \tilde{\alpha}) - \omega_2) = 0$. We conclude from the lemma below that $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = 0$ for $0 < i < 2p-3$ and further that $\mathrm{H}^{2p-3}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = 0$ unless $\lambda = (p-n)\tilde{\alpha} = 7\tilde{\alpha}$.

Lemma . *Suppose that Φ is of type A_4 with $p = 11$. Then*

(a) $\sum_{u \in W} (-1)^{\ell(u)} P_6(u \cdot (6\omega_2) - \omega_2) = 0$,
(b) $\sum_{u \in W} (-1)^{\ell(u)} P_7(u \cdot (6\omega_2 + \tilde{\alpha}) - \omega_2) = 0$.

Proof. (a) Observe that $5\omega_2 = 3\alpha_1 + 6\alpha_2 + 4\alpha_3 + 2\alpha_4 = 3\epsilon_1 + 3\epsilon_2 - 2\epsilon_3 - 2\epsilon_4 - 2\epsilon_5$. For $u \cdot (6\omega_2) - \omega_2$ to be a sum of positive roots one needs $u(5\omega_2)$ to be a sum of positive roots. This is the case if and only if either $u(5\omega_2) = 5\omega_2$ or $u(5\omega_2) = 3\alpha_1 + \alpha_2 + 4\alpha_3 + 2\alpha_4$. Let us consider the second case. Here u is an element of the co-set $s_2 \cdot \mathrm{Stab}_W(\omega_2)$. Note that for any $u \in s_2 \cdot \mathrm{Stab}_W(\omega_2)$ the expression $-u \cdot 0$ contains some positive multiple of the root α_2. Therefore $u \cdot (6\omega_2) - \omega_2 = 3\alpha_1 + 4\alpha_3 + 2\alpha_4 + u \cdot 0$ is not a sum of positive roots. It suffices therefore to look at $\sum_{u \in \mathrm{Stab}_W(\omega_2)} (-1)^{\ell(u)} P_6(5\omega_2 + u \cdot 0)$. A straightforward but tedious calculation now shows that $\sum_{u \in \mathrm{Stab}_W(\omega_2)} (-1)^{\ell(u)} P_6(5\omega_2 + u \cdot 0) = 0$.

(b) As in part (a) for $u \cdot (6\omega_2 + \tilde{\alpha}) - \omega_2$ to be the sum of positive roots one needs either $u \in \mathrm{Stab}_W(\omega_2)$ or $u \in s_2 \cdot \mathrm{Stab}_W(\omega_2)$. However, the second case results in weights that cannot be written as sums of 7 positive roots. As in part (a) it suffices therefore to look at $\sum_{u \in \mathrm{Stab}_W(\omega_2)} (-1)^{\ell(u)} P_7(6\omega_2 + u \cdot \tilde{\alpha})$ which can be shown to be zero.

Both parts of the lemma can also be readily checked by using computer software such as MAGMA [BC, BCP]. □

6.13 Summary for $r = 1$. The following theorem addresses (1.1.1) and (1.1.2) for type A_n when $r = 1$.

Theorem . *Suppose Φ is of type A_n with $n \geq 2$. Suppose further that $p > n + 1$.*

(a) *(Generic case) If $p > n + 2$ and $n > 3$, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p - 3$;
  (ii) $\mathrm{H}^{2p-3}(G(\mathbb{F}_p), k) \cong k$.
(b) *If $p = n + 2$, then*

(i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < p-2$*;*
(ii) $\mathrm{H}^{p-2}(G(\mathbb{F}_p), k) \cong k \oplus k$.

(c) *If* $n = 2$ *and* 3 *divides* $p-1$*, then*
(i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-6$*;*
(ii) $\mathrm{H}^{2p-6}(G(\mathbb{F}_p), k) \cong k \oplus k$.

(d) *If* $n = 2$ *and* 3 *does not divide* $p-1$*, then*
(i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-3$*;*
(ii) $\mathrm{H}^{2p-3}(G(\mathbb{F}_p), k) \cong k$.

(e) *If* $n = 3$ *and* $p > 5$*, then*
(i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-6$*;*
(ii) $\mathrm{H}^{2p-6}(G(\mathbb{F}_p), k) \cong k$.

Proof. Let $\lambda = p\mu + w \cdot 0 \in X(T)_+$ for $\mu \in X(T)_+$ and $w \in W$. Based on the discussion in Section 2, our goal has been to determine the least $i > 0$ such that

$$\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0.$$

According to Proposition 6.1(a), we know that the weight $\lambda = p\tilde{\alpha} - n\tilde{\alpha}$ gives a non-zero cohomology class in degree $2p-3$.

By Proposition 3.4(b), if $\langle \mu, \tilde{\alpha}^\vee \rangle \geq 2$, then $i \geq 2p-3$. Hence, the only way to obtain a smaller i is for μ to be a fundamental weight. That case has been dealt with in previous sections, from which parts (a)(i), (b), (c), (d)(i), and (e) follow. It remains to show parts (a)(ii) and (d)(ii). From Proposition 6.4, $\lambda = p\tilde{\alpha} - n\tilde{\alpha} = p(\omega_1 + \omega_n)$ is the only weight with $\mathrm{H}^{2p-3}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$. The result follows by Proposition 6.1(a) and Theorem 2.8(A). □

6.14 Results for $r > 1$**.** The following theorem addresses (1.1.1) and (1.1.2) for type A_n when $r > 1$ and $p > 2n-2$. For $n > 3$, a generic vanishing bound of degree $r(2p-3)$ can be observed.

For $n+2 \leq p \leq 2(n+1)$, the methods employed in this paper should allow one to obtain precise vanishing bounds. However, given the number of special cases encountered in the $r = 1$ case, one would expect even more non-generic behavior for $r > 1$. For example, it is easily seen that in the case $p = n+2$ non-vanishing already occurs in degree $r(p-2)$, i.e., $\mathrm{H}^{r(p-2)}(G(\mathbb{F}_q), k) \neq 0$. To give a complete answer many case-by-case arguments will be necessary, most of them rather lengthy and intricate. For brevity we limit ourselves here to the case where p is larger than twice the Coxeter number.

Theorem . *Suppose* Φ *is of type* A_n *with* $n \geq 2$ *and* $p > 2(n+1)$*. Then*

(a) *(Generic case) If* $n > 3$*, then*
(i) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < r(2p-3)$*;*
(ii) $\mathrm{H}^{r(2p-3)}(G(\mathbb{F}_q), k) \cong k$.

(b) *If* $n = 2$ *and* 3 *divides* $q-1$*, then*
(i) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < r(2p-6)$*;*
(ii) $\mathrm{H}^{r(2p-6)}(G(\mathbb{F}_q), k) \cong k \oplus k$.

(c) *If* $n = 2$ *and* 3 *does not divide* $q-1$*, then*
(i) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < r(2p-3)$*;*

(ii) $\mathrm{H}^{r(2p-3)}(G(\mathbb{F}_q),k)\cong k$.

(d) *If* $n=3$, *then*

(i) $\mathrm{H}^i(G(\mathbb{F}_q),k)=0$ *for* $0<i<r(2p-6)$;

(ii) $\mathrm{H}^{r(2p-6)}(G(\mathbb{F}_q),k)\cong k$.

Proof. From Remark 6.2, for $\lambda=p\delta+u\cdot 0$ with δ in the root lattice, the following holds

$$\operatorname{Ext}^i_G(V(\lambda)^{(r)},H^0(\lambda))\cong\begin{cases}0 & \text{if } i<2p-3,\\ 0 & \text{if } i=r(2p-3) \text{ and } \lambda\neq(p-n)(\omega_1+\omega_n),\\ k & \text{if } i=r(2p-3) \text{ and } \lambda=(p-n)(\omega_1+\omega_n).\end{cases}\tag{6.14.1}$$

From now on assume that $\lambda=p\delta+u\cdot 0$ and that δ not in the root lattice. Our goal is to obtain results like (6.14.1) for this situation.

Assume further that $\operatorname{Ext}^i_G(V(\lambda)^{(r)},H^0(\lambda))\neq 0$. By Proposition 4.2, there exists a sequence of non-zero weights $\lambda=\gamma_0,\gamma_1,\dots,\gamma_{r-1},\gamma_r=\lambda\in X(T)_+$ and nonnegative integers $l_1,l_2,\dots,l_r$ such that

(i) $i=\sum_{j=1}^r l_j$,

(ii) $\operatorname{Ext}^{l_j}_G(V(\gamma_j)^{(1)},H^0(\gamma_{j-1}))\neq 0$, for $1\le j\le r$, and

(iii) $\gamma_j=p\delta_j+u_j\cdot 0$ for some $u_j\in W$ and nonzero $\delta_j\in X(T)_+$.

Note that none of the δ_j are contained in the root lattice.

Next we apply our discussion in Section 6.3 to the pair of weights γ_j,γ_{j-1} and obtain from (6.3.2)

$$l_j\ge 2pM_{m(\delta_j)}(\delta_j)+2M_{m(\delta_j)}(u_j\cdot 0)-2M_{m(\delta_{j-1})}(\delta_{j-1})-M_{m(\delta_{j-1})}(u_{j-1}\cdot 0).\tag{6.14.2}$$

It follows from $\delta_r=\delta_0$ and $u_r=u_0$ that

$$\begin{aligned} i=\sum_{j=1}^r l_j &\ge \sum_{j=1}^r 2pM_{m(\delta_j)}(\delta_j)+2M_{m(\delta_j)}(u_j\cdot 0)-2M_{m(\delta_{j-1})}(\delta_{j-1})-M_{m(\delta_{j-1})}(u_{j-1}\cdot 0)\\ &= \sum_{j=1}^r 2(p-1)M_{m(\delta_j)}(\delta_j)+M_{m(\delta_j)}(u_j\cdot 0)\\ &\ge \sum_{j=1}^r 2(p-1)M_{m(\delta_j)}(\delta_j)+M_{m(\delta_j)}(-2\rho),\end{aligned}$$

where the last inequality follows from Observation 6.3(f). We define

$$d_j:=l_j+2M_{m(\delta_{j-1})}(\delta_{j-1})-2M_{m(\delta_j)}(\delta_j)-M_{m(\delta_j)}(u_j\cdot 0)+M_{m(\delta_{j-1})}(u_{j-1}\cdot 0).\tag{6.14.3}$$

Then $i=\sum_{j=1}^r d_j$ and $d_j\ge 2(p-1)M_{m(\delta_j)}(\delta_j)+M_{m(\delta_j)}(-2\rho)$.

In order to show vanishing up to the claimed degrees it is sufficient to show that $\sum_{j=1}^r d_j\ge r(2p-3)$, ($r(2p-6)$, respectively). We will actually show that strict inequalities hold in all but very few special cases. These special cases will yield statements (b)(ii) and (d)(ii) of the theorem.

According to Observation 6.3(e), we know that $M_{m(\delta_j)}(-2\rho)=-N_{m(\delta_j)}$. Moreover, using the arguments in the proofs of Propositions 6.2, 6.3, and 6.4, one obtains the following

bounds (recall that δ_j does not lie in the root lattice):

$$d_j \geq \begin{cases} \left[2\left(\frac{p-1}{n+1}\right)-1\right](n+1-m(\delta_j))m(\delta_j) & \text{if } \delta_j = \omega_{m(\delta_j)}, \\ 2(p-1)+2(n-1)\left[\frac{p-1}{n+1}-1\right] & \text{else.} \end{cases} \tag{6.14.4}$$

Note that, as observed in Section 6.4, the second expression is strictly larger than $2p-3$ under the assumptions on p and n. Therefore, the only way to possibly obtain a d_j with $d_j \leq 2p-3$ ($2p-6$, respectively) occurs when δ_j is a single fundamental weight.

Case 1: $\delta_j \in \{\omega_2, ..., \omega_{n-1}\}, n > 2$.

The expression $\left[2\left(\frac{p-1}{n+1}\right)-1\right](n+1-m(\delta_j))m(\delta_j)$ attains a minimum when $m(\delta_j) = 2$ or $n-1$ (see Section 6.8). Hence,

$$\begin{aligned} d_j &\geq 2(p-1)+(2n-6)\left(\frac{p-1}{n+1}\right)-2n+2 \\ &\geq 2(p-1)+2(2n-6)-2n+2 \\ &= 2(p-1)+2n-10. \end{aligned}$$

If $n \geq 5$ one obtains $d_j \geq 2p-2 > 2p-3$. For $n = 4$ one can show that $d_j > 2p-3$ whenever $p > 13$. We will discuss the case $n = 4$ and $p \in \{11, 13\}$ separately later. For $n = 3$ one obtains $d_j \geq 2p-6$.

Case 2: $\delta_j = \omega_1$ or ω_n.

Here the above methods produce the lower bound

$$d_j \geq 2(p-1)-2\left(\frac{p-1}{n+1}\right)-n, \tag{6.14.5}$$

which is not sufficient. Other methods have to be applied. We will distinguish two cases.

Case 2.1: δ_{j-1} is a fundamental weight.

We apply Lemma 6.9 to obtain from equation (6.9.1)

$$l_j \geq (p-k)n-(n+1-k)k-(n+1-s)s+\ell(u_{j-1}),$$

where $k = \ell(u_j) = -M_1(u_j \cdot 0)$. By (6.14.3)

$$d_j \geq l_j + 2M_s(\omega_s) - 2M_1(\omega_1) - M_1(u_j \cdot 0) + M_s(u_{j-1} \cdot 0).$$

From Observation 2.2(B) (see Section 6.3), we know that $\ell(u_{j-1}) \geq -M_s(u_{j-1} \cdot 0)$ and clearly $M_s(\omega_s) \geq M_1(\omega_1)$. This yields

$$d_j \geq (p-k)n-(n+1-k)k+k-(n+1-s)s = (p-k)n-(n-k)k-(n+1-s)s.$$

As a function of k, the above attains its minimum at $k = n$. Hence

$$d_j \geq (p-n)n-(n+1-s)s = 2p-3+(n-2)p-n^2-(n+1-s)s+3.$$

Using the assumption that $p \geq 2n+3$, one obtains

$$\begin{aligned} d_j &\geq 2p-3+(n-2)(2n+3)-n^2-(n+1-s)s+3 \\ &= 2p-3+n^2-n-(n+1-s)s-3. \end{aligned}$$

As an integer function of s, the above attains its minimum at

$$s = \begin{cases} (n+1)/2 & \text{for } n \text{ odd,} \\ n/2 & \text{for } n \text{ even.} \end{cases}$$

One concludes that

$$d_j \geq \begin{cases} 2p-3+\frac{3(n-1)^2}{4}-4 & \text{for } n \text{ odd,} \\ 2p-3+\frac{3n(n-2)}{4}-3. & \text{for } n \text{ even.} \end{cases}$$

For $n > 3$ this yields $d_j > 2p-3$, for $n = 3$ this yields $d_j \geq 2p-4$, and for $n = 2$ this yields $d_j \geq 2p-6$.

Case 2.2: δ_{j-1} is not a fundamental weight.

Here we will show that $d_j + d_{j-1} \geq 2(2p-3)$ (in the generic case). If $j = 1$ we set $d_0 = d_r$. Note that the following argument makes sense because $\gamma_0 = \gamma_r$. Recall that none of the δ_j are contained in the root lattice. Therefore, (6.14.4) yields

$$d_{j-1} \geq 2(p-1) + 2(n-1)\left[\frac{p-1}{n+1} - 1\right]. \tag{6.14.6}$$

Adding (6.14.5) and (6.14.6) produces

$$\begin{aligned} d_j + d_{j-1} &\geq 2(2p-3) + 2 + 2(n-2)\left(\frac{p-1}{n+1}\right) - 3n + 2 \\ &\geq 2(2p-3) + 4(n-2) - 3n + 4 \\ &= 2(2p-3) + n - 4. \end{aligned}$$

If $n \geq 5$ one obtains $d_j + d_{j-1} > 2(2p-3)$. The same holds for $n = 4$ and $p > 11$ (since the second inequality above is in fact strict). For $n = 4$ and $p = 11$, one has $d_j + d_{j-1} = 2(2p-3)$ only if $\delta_{j-1} \in \{\omega_1+\omega_2, \omega_1+\omega_3, \omega_4+\omega_2, \omega_4+\omega_3\}$. We will discuss this case later. For all others weights we get a strict inequality. For $n = 2$ and $n = 3$ it follows that $d_j + d_{j-1} \geq 2(2p-4)$.

Assume now that $n > 3$. If $n = 4$ assume in addition that $p > 13$. From above, we see that $d_j > 2p-3$ unless $\delta_j = \omega_1$ or ω_n and δ_{j-1} is not a fundamental weight. If $d_j > 2p-3$ for each $1 \leq j \leq r$, we have

$$i \geq \sum_{j=1}^{r} d_j > r(2p-3)$$

and vanishing for positive degrees up to $r(2p-3)$.

Suppose now that $d_j \leq 2p-3$ for some j. Recall that none of δ_j are assumed to be in the root lattice. Let t be the largest such j and suppose that $t > 1$. Then

$$i \geq \sum_{j=1}^{r} d_j = \sum_{t+1}^{r} d_j + d_t + d_{t-1} + \sum_{j=1}^{t-2} d_j > (r-t)(2p-3) + 2(2p-3) + \sum_{j=1}^{t-2} d_j.$$

Consider the remaining sum $\sum_{j=1}^{t-2} d_j$, again identify the largest j with $d_j \leq 2p-3$, and repeat this decomposition. Continuing in this manner the claim follows except possibly if we reach a point when the largest j with $d_j \leq 2p-3$ is $j=1$. So we are done if $d_1 > 2p-3$.

Suppose now that $\delta_1 = \omega_1$ or ω_n and δ_0 is not a fundamental weight so that we could have $d_1 \leq 2p-3$. Note that since δ_1 is a fundamental weight, $d_2 > 2p-3$. Recall that $\delta_0 = \delta_r$. Therefore, we can use the Case 2.2 argument to show that $d_1 + d_r > 2(2p-3)$. Further, since $d_2 > 2p-3$, the above argument can successfully be used to show that $\sum_{j=2}^{r-1} d_j > (r-2)(2p-3)$. Hence,

$$i \geq d_1 + d_r + \sum_{j=2}^{r-1} d_j > 2(2p-3) + (r-2)(2p-3) = r(2p-3).$$

One concludes that (6.14.1) holds for any dominant weight λ as long as $n \geq 5$ or $n=4$ and $p > 13$.

For $n=3$ the above cases show that $i = \sum_{i=1}^{r} d_j \geq r(2p-6)$ with equality only in the case that all $\delta_j = \omega_2$. Using the arguments in Section 6.11 one concludes that for weights $\mu_1 = p\omega_2 + w_1 \cdot 0$ and $\mu_2 = p\omega_2 + w_2 \cdot 0$

$$\operatorname{Ext}^i_G(V(\mu_2)^{(1)}, H^0(\mu_1)) = \begin{cases} 0 & \text{if } 0 < i < 2p-6, \\ 0 & \text{if } i = 2p-6 \text{ and not both } \mu_1, \mu_2 \text{ are equal to } (p-4)\omega_2, \\ k & \text{if } i = 2p-6 \text{ and } \mu_1 = \mu_2 = (p-4)\omega_2. \end{cases}$$

An argument similar to those in Sections 5.4 and 6.2 now yields

$$\operatorname{Ext}^i_G(V(\lambda)^{(r)}, H^0(\lambda)) = \begin{cases} 0 & \text{if } 0 < i < r(2p-6), \\ 0 & \text{if } i = r(2p-6) \text{ and } \lambda \neq (p-4)\omega_2, \\ k & \text{if } i = r(2p-6) \text{ and } \lambda = (p-4)\omega_2. \end{cases} \tag{6.14.7}$$

If $n=2$ and 3 does not divide p^r-1 then $\operatorname{Ext}^i_G(V(\lambda)^{(r)}, H^0(\lambda)) \neq 0$ forces λ and all the γ_j to be in the root lattice. By Remark 6.2 one obtains (6.14.1).

If $n=2$ and 3 divides p^r-1, one concludes from the above discussion that $i = \sum_{i=1}^{r} d_j \geq r(2p-6)$ with equality only in the case that all $\delta_j \in \{\omega_1, \omega_2\}$. A direct computation using Lemma 6.9 now shows that for weights $\mu_1 = p\omega_r + w_1 \cdot 0$ and $\mu_2 = p\omega_s + w_2 \cdot 0$ with $r, s \in \{1,2\}$

$$\dim \operatorname{Ext}^i_G(V(\mu_2)^{(1)}, H^0(\mu_1)) = \begin{cases} 0 & \text{if } 0 < i < 2p-6, \\ 0 & \text{if } i = 2p-6 \\ & \text{and not both } \mu_1, \mu_2 \text{ are in } \{(p-3)\omega_i \mid i=1,2\}, \\ \delta_{rs} & \text{if } i = 2p-6, \text{ both } \mu_1, \mu_2 \in \{(p-3)\omega_i \mid i=1,2\} \\ & \text{and } p \equiv 1 \bmod 3, \\ 1-\delta_{rs} & \text{if } i = 2p-6, \text{ both } \mu_1, \mu_2 \in \{(p-3)\omega_i \mid i=1,2\} \\ & \text{and } p \equiv -1 \bmod 3. \end{cases}$$

Arguments like the ones in Sections 5.4 and 6.2 now yield

$$\text{(6.14.8)} \qquad \operatorname{Ext}^i_G(V(\lambda)^{(r)}, H^0(\lambda)) = \begin{cases} 0 & \text{if } 0 < i < r(2p-6), \\ 0 & \text{if } i = r(2p-6) \text{ and } \lambda \neq (p-3)\omega_i,\ i \in \{1,2\}, \\ k & \text{if } i = r(2p-6) \text{ and } \lambda = (p-3)\omega_i,\ i \in \{1,2\}. \end{cases}$$

Note that in the case when 3 divides $p-1$ all δ_j are the same, while in the case that 3 divides $p+1$ the δ_j alternate between ω_1 and ω_2.

That leaves only the cases $n=4$ and $p=11$ or 13. We will show that (6.14.1) holds in these cases. We discuss $p=13$ first. Looking at Cases 1 through 2.2 one can see that $d_j \leq 2p-3$ occurs when $\delta_j = \omega_2$ or its dual weight ω_3. We will discuss the case $\delta_j = \omega_2$. We distinguish two cases, namely, δ_{j-1} being the sum of at least two fundamental weights and δ_{j-1} being a fundamental weight. In the first situation one can use an argument similar to Case 2.2 to show that $d_j + d_{j-1} > 2(2p-3)$. We leave the details to the interested reader. In the second situation the weight $\delta_{j-1} = \omega_1$, because $p\delta_j - \delta_{j-1}$ has to be an element of the root lattice. In our series of estimates for i and l_j in (6.3.1), (6.3.2), and (6.14.2), we replaced $-2M_{m(\delta_j)}(\delta_{j-1})$ by $-2M_{m(\delta_{j-1})}(\delta_{j-1})$. The former clearly being greater than or equal to the latter. Without this substitution one can obtain a better estimate for d_j, namely $2(p-1)M_{m(\delta_j)}(\delta_j) + M_{m(\delta_j)}(-2\rho) + 2M_{m(\delta_{j-1})}(\delta_{j-1}) - 2M_{m(\delta_j)}(\delta_{j-1})$. In our special case this yields $d_j \geq 2 \cdot 12 \cdot 6/5 - 6 + 2 \cdot 4/5 - 2 \cdot 3/5 > 23$, as required.

When $n=4$ and $p=11$ we need to revisit Case 1 with $\delta_j = \omega_2$ or ω_3. We proceed as in the $p=13$ case. If δ_{j-1} is the sum of more than one fundamental weight an argument similar to Case 2.2 will show that $d_j + d_{j-1} > 2(2p-3)$. If δ_{j-1} is a single fundamental weight then the prime forces $\delta_{j-1} = \delta_j$. Here Lemma 6.12 implies $d_j > 2p-3$. Details are left to the reader. The last remaining situation is when $\delta_j = \omega_1$ or ω_4. Note that equality holds in (6.14.6) if and only if $\delta_{j-1} \in \{\omega_1+\omega_2, \omega_1+\omega_3, \omega_4+\omega_2, \omega_4+\omega_3\}$. In any of these four cases the height argument used in Case 2.1 yields $d_j = 2p-4$. From (6.14.6) one obtains $d_{j-1} = 2p+4$. Hence $d_j + d_{j-1} > 2(2p-3)$ and (6.14.1) holds for $n \geq 4$.

The weights $\lambda = (p-n)\tilde{\alpha}$ as well as $(p-4)\omega_2$ for $n=3$ are the lowest non-zero dominant weights in their linkage classes. Theorem 2.8(A), (6.14.1), and (6.14.7) imply parts (a), (c) and (d). Similarly, a slight variation of Theorem 2.8(A) and (6.14.8) yield part (b).

□

6.15 The General Linear Group $GL_n(\mathbb{F}_q)$. For convenience we assumed throughout the paper that G is a simple algebraic group. However, most results can be generalized to split reductive groups such as $GL_n(k)$. In particular Sections 2.3 through 2.8 and Formula (3.2.1) are valid for this larger set of groups. One can therefore argue as in Sections 6.1 and 6.2 and obtain the following:

Proposition . *Suppose $n \geq 1$, $r \geq 1$ and $p \geq n+2$. Let $\lambda = (p-n)\tilde{\alpha}$, $\tilde{\alpha}$ being the maximal positive root. Then*

(a) $\mathrm{H}^{r(2p-3)}(GL_n(k), H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \cong k$;
(b) $\mathrm{H}^{r(2p-3)}(GL_n(\mathbb{F}_q), k) \neq 0$.

In [FP, Appendix] Friedlander and Parshall found the sharp bound $r(2p-3)$ for the Borel subgroup $B(\mathbb{F}_q)$ of $GL_n(\mathbb{F}_q)$ for all odd q. Using the fact that the restriction map

$\mathrm{H}^i(GL_n(\mathbb{F}_q),k) \to \mathrm{H}^i(B(\mathbb{F}_q),k)$ is injective one obtains the vanishing range of $0 < i < r(2p-3)$ for the cohomology of $GL_n(\mathbb{F}_q)$. One can now combine this with the aforementioned proposition to obtain the following theorem which verifies the conjecture in Barbu [B, Conjecture 4.11] for $p \geq n+2$.

Theorem . *Suppose $n \geq 1$.*

(a) *If q is odd, then $\mathrm{H}^i(GL_n(\mathbb{F}_q),k) = 0$ for $0 < i < r(2p-3)$;*
(b) *If $p \geq n+2$, then $\mathrm{H}^{r(2p-3)}(GL_n(\mathbb{F}_q),k) \neq 0$.*

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